Flavor neutrino states in quantum field theory

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[1] V. Gribov and B. Pontecorvo, Phys. Lett. B **28**, 493 (1969).
[2] L.N. Chang and N.P. Chang, Phys. Rev. Lett. **45**, 1540 (1980).
[3] P.T. Mannheim, Phys. Rev. D **37**, 1935 (1988).
[4] C. Giunti, C.W. Kim and U.W. Lee, Phys. Rev. D **45**, 2414 (1992).
[5] C. Giunti, C.W. Kim, J.A. Lee and U.W. Lee, Phys. Rev. D **48**, 4310 (1993).
[6] M.Blasone and G.Vitiello, Ann. Phys. **244**, 283 (1995).
[7] K.C. Hannabuss and D.C. Latimer, J. Phys. A **33**, 1369 (2000).

The problem of flavor states

Consider the process $P_I \rightarrow P_F + l_\sigma^+ + \nu_\sigma$. Consider the S-matrix element

$$\langle \nu_\sigma l_\sigma^+ P_F | S | P_I \rangle$$

What is definition of $|\nu_\sigma\rangle$? Field mixng transformation

$$\nu_\sigma(x) \; = \; \sum_j U_{\sigma\, j} \nu_j(x)$$

between flavor fields ν_σ and mass fields ν_j. U is the mixing matrix. In the two-flavor case it is parametrized as:

$$U \; = \; \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{1}$$

Mass eigenstates

Fields with definite masses can be expanded as:

$$\nu_i(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k},r} \left[u^r_{\mathbf{k},i}(t)\alpha^r_{\mathbf{k},i} \ + \ v^r_{-\mathbf{k},i}(t)\beta^{r\dagger}_{-\mathbf{k},i} \right] e^{i\mathbf{k}\cdot\mathbf{x}} \,, \quad i = 1,2$$

A mass-eigenstate neutrino is defined as:

$$|\nu^r_{\mathbf{k},i}\rangle \ = \ \alpha^{r\dagger}_{\mathbf{k},i}|0\rangle_{1\,2}$$

mass vacuum is defined by:

$$\alpha^r_{\mathbf{k},i}|0\rangle_{1\,2} \ = \ \beta^r_{\mathbf{k},i}|0\rangle_{1\,2} \ = \ 0$$

Pontecorvo flavor states

Pontecorvo states[8]:

$$
\begin{aligned}
|\nu^r_{\mathbf{k},e}\rangle_P &= \cos\theta\, |\nu^r_{\mathbf{k},1}\rangle + \sin\theta\, |\nu^r_{\mathbf{k},2}\rangle \\
|\nu^r_{\mathbf{k},\mu}\rangle_P &= -\sin\theta\, |\nu^r_{\mathbf{k},1}\rangle + \cos\theta\, |\nu^r_{\mathbf{k},2}\rangle
\end{aligned}
$$

Consider the amplitude of the neutrino detection process $\nu_\sigma + X_i \to e^- + X_f$:

$$
\langle e^s_{\mathbf{q},-}|\bar{e}(x)\,\gamma^\mu\,(1-\gamma^5)\,\nu_e(x)|\nu^r_{\mathbf{k},\sigma}\rangle_P\, h_\mu(x) \;\not\propto\; \delta_{\sigma e}
$$

h_μ are the matrix elements of the X part. PROBLEM: Neutrino flavor is detected by identifying the charged-lepton.

[8] S.M. Bilenky and B. Pontecorvo, Phys. Rep. **41**, 225 (1978)

WI Lagrangian: Flavor Basis

Lepton (W) sector os Standard Model (after SSB). Free Lagrangian:

$$\mathcal{L}_0 = \sum_{\sigma,\rho=e,\mu} \left[\overline{\nu}_\sigma \left(i\gamma_\mu \partial^\mu - M_\nu^{\sigma\rho}\right)\nu_\rho + \bar{l}_\sigma \left(i\gamma_\mu \partial^\mu - M_l^{\sigma\rho}\right) l_\rho\right]$$

where $l_e \equiv e$, $l_\mu \equiv \mu$, and:

$$M_\nu = \begin{bmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{bmatrix} \quad ; \qquad M_l = \begin{bmatrix} \tilde{m}_e & 0 \\ 0 & \tilde{m}_\mu \end{bmatrix}$$

Interacting part:

$$\mathcal{L}_{int} = \frac{g}{2\sqrt{2}} \sum_{\sigma=e,\mu} \left[W_\mu^+(x)\,\overline{\nu}_\sigma\,\gamma^\mu\left(1-\gamma^5\right) l_\sigma + h.c.\right]$$

WI Lagrangian: Mass Basis

Kinetic part diagonalized by mixing transformation
$(\tan 2\theta \;=\; 2m_{e\mu}/(m_e - m_\mu))$

$$\mathcal{L}_0 \;=\; \sum_{j=1,2} \overline{\nu}_j \left(i\gamma_\mu \partial^\mu - m_j\right) \nu_j \;+\; \sum_{\sigma=e,\mu} \bar{l}_\sigma \left(i\gamma_\mu \partial^\mu - \tilde{m}_\sigma\right) l_\sigma$$

where:

$$\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta \\ \sin^2\theta & \cos^2\theta \end{bmatrix} \begin{bmatrix} m_e \\ m_\mu \end{bmatrix}$$

Interacting part in no-more diagonal:

$$\mathcal{L}_{int} \;=\; \frac{g}{2\sqrt{2}} \sum_{\sigma=e,\mu} \sum_{j=1,2} \left[W_\mu^+(x)\, \overline{\nu}_j \, U_{j\sigma}^* \, \gamma^\mu \left(1 - \gamma^5\right) l_\sigma + h.c. \right]$$

Weak Process states

Then. mass neutrinos mix in the interaction. If mass neutrinos are taken as physical ⇒ Weak Process (production) states[9]:

$$|\nu_\sigma^r\rangle_{WP} \;\equiv\; \sum_j \mathcal{A}_{\sigma j}\,|\nu_j^r\rangle$$

where

$$\mathcal{A}_{\sigma j} = \langle \nu_j l_\sigma^+ P_F|S|P_I\rangle$$

Flavor states definition depends on the process. These present the same problem as Pontecorvo states ⇒ flavor violation (at tree level) in the production vertex.

[9] C. Giunti and C.W. Kim, *Fundamentals of Neutrino Physics and Astrophysics* (Oxford Univ. Press, 2007)

Lepton number conservation

The above Lagrangian is invariant under the global $U(1)$ transformation:

$$\begin{aligned} e(x) &\to e^{i\alpha}\, e(x)\,, \qquad & \nu_e(x) &\to e^{i\alpha}\, \nu_e(x) \\ \mu(x) &\to e^{i\alpha}\, \mu(x)\,, \qquad & \nu_\mu(x) &\to e^{i\alpha}\, \nu_\mu(x) \end{aligned}$$

Noether's charge:

$$Q_l^{tot} = \sum_{\sigma=e,\mu} Q_\sigma^{tot}(t)\,, \quad Q_\sigma^{tot}(t) = Q_{\nu_\sigma}(t) + Q_\sigma$$

where

$$\begin{aligned} Q_e &= \int \mathrm{d}^3\mathbf{x}\, e^\dagger(x) e(x)\,, \qquad & Q_{\nu_e}(t) &= \int \mathrm{d}^3\mathbf{x}\, \nu_e^\dagger(x)\nu_e(x) \\ Q_\mu &= \int \mathrm{d}^3\mathbf{x}\, \mu^\dagger(x) \mu(x)\,, \qquad & Q_{\nu_\mu}(t) &= \int \mathrm{d}^3\mathbf{x}\, \nu_\mu^\dagger(x)\nu_\mu(x) \end{aligned}$$

Conservation of the total lepton number

Weak production and Flavor eigenstates

Because of mixing:

$$\left[Q_\sigma^{tot}(t)\,,\,\mathcal{L}_0(x)\right] \neq 0\,, \qquad \sigma = e, \mu$$

However

$$\left[Q_\sigma^{tot}(t)\,,\,\mathcal{L}_{int}(x)\right] \;=\; 0$$

Leptons are produced (at tree level) as flavor eigenstates[10]

Neutrino *flavor eigenstates* are not the same as *mass eigenstates*.

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[10] M. Blasone, A. Capolupo, C. R. Ji and G. Vitiello, Nucl. Phys. B Proc. Suppl. **188**, 37-39 (2009).

Flavor states and flavor Fock space

Mixing generator

Mixing transformation can be rewritten as

$$\begin{aligned}
\nu_e(x) &= G_\theta^{-1}(t)\nu_1(x)\, G_\theta(t) \\
\nu_\mu(x) &= G_\theta^{-1}(t)\, \nu_2(x) G_\theta(t)
\end{aligned}$$

Mixing generator:

$$G_\theta(t) = \exp\left[\theta \int d^3\mathbf{x}\, \left(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x)\right)\right]$$

Decomposition of the mixing generator (1)

Mixing generator can be decomposed as[11]:

$$G_\theta \;=\; B(\Theta_1,\Theta_2)\;\; R(\theta)\;\; B^{-1}(\Theta_1,\Theta_2)$$

where $B(\Theta_1,\Theta) \equiv B_1(\Theta_1)\, B_2(\Theta_2)$,

$$R(\theta) \equiv \exp\left\{\theta \sum_{\mathbf{k},r} \Big[\left(\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r}_{\mathbf{k},2} + \beta^{r\dagger}_{-\mathbf{k},1}\beta^{r}_{-\mathbf{k},2}\right) e^{i\psi_{\mathbf{k}}} - h.c.\Big]\right\}$$

$$B_i(\Theta_i) \equiv \exp\Big\{\sum_{\mathbf{k},r} \Theta_{\mathbf{k},i}\; \epsilon^r \Big[\alpha^{r}_{\mathbf{k},i}\beta^{r}_{-\mathbf{k},i} e^{-i\phi_{\mathbf{k},i}} - \beta^{r\dagger}_{-\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},i} e^{i\phi_{\mathbf{k},i}}\Big]\Big\}, \quad i = 1,2$$

and $\Theta_{\mathbf{k},i} = 1/2\ \cot^{-1}(|\mathbf{k}|/m_i), \quad \psi_{\mathbf{k}} = (\omega_{\mathbf{k},1} - \omega_{\mathbf{k},2})t, \quad \phi_{\mathbf{k},i} = 2\omega_{\mathbf{k},i}t.$

[11] M.Blasone, M.V.Gargiulo and G.Vitiello, Phys. Lett.B **761**, 104 (2016)

Decomposition of the mixing generator (2)

$B_i(\Theta_{\mathbf{k},i})$, $i = 1, 2$ are Bogoliubov transformations which induces a mass shift and $R(\theta)$ is a rotation.

Their action on the mass vacuum is:

$$
\begin{aligned}
|\widetilde{0}\rangle_{1,2} &\equiv B^{-1}(\Theta_1, \Theta_2)|0\rangle_{1,2} \\
&= \prod_{\mathbf{k},r} \Big[\cos\Theta_{\mathbf{k},i} + \epsilon^r \sin\Theta_{\mathbf{k},i} \alpha^{r\dagger}_{\mathbf{k},i} \beta^{r\dagger}_{-\mathbf{k},i} \Big] |0\rangle_{1,2} \\
R^{-1}(\theta)|0\rangle_{1,2} &= |0\rangle_{1,2}
\end{aligned}
$$

A rotation of fields is not a rotation at the level of creation and annihilation operators!

Flavor Vacuum

Flavor vacuum is defined by[12]:

$$|0\rangle_{e,\mu} \equiv G_\theta^{-1}(0)\ |0\rangle_{1,2}$$

In the infinite volume limit:

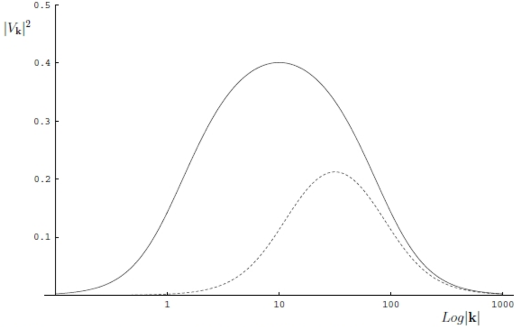
$$\lim_{V\to\infty}\ {}_{1,2}\langle 0|0\rangle_{e,\mu} = \lim_{V\to\infty}\ e^{V\int \frac{d^3\mathbf{k}}{(2\pi)^3}\ \ln\left(1-\sin^2\theta\,|V_\mathbf{k}|^2\right)^2} = 0$$

where

$$|V_\mathbf{k}|^2 \ \equiv\ \sum_{r,s} |\ v^{r\dagger}_{-\mathbf{k},1} u^s_{\mathbf{k},2}\ |^2 \neq 0 \quad for \quad m_1 \neq m_2$$

[12]M.Blasone and G.Vitiello, Ann. Phys. **244**, 283 (1995)

Vacuum condensate



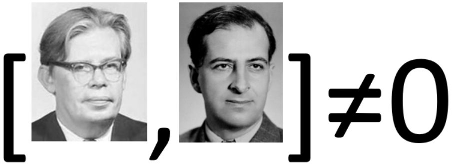
Solid line: $m_1 = 1$, $m_2 = 100$; Dashed line: $m_1 = 10$, $m_2 = 100$.

- Condensation density: ${}_{e,\mu}\langle 0|\alpha^{r\dagger}_{\mathbf{k},i}\alpha_{\mathbf{k},i}|0\rangle_{e,\mu} = \sin^2\theta\,|V_{\mathbf{k}}|^2$, with $i = 1, 2$. Same result for antiparticles.

- $|V_{\mathbf{k}}|^2 \simeq \frac{(m_2-m_1)^2}{4k^2}$ for $k \gg \sqrt{m_1 m_2}$.

Bogoliubov vs Pontecorvo

- $\left[B(m_1, m_2)\ ,\ R^{-1}(\theta)\right] \neq 0$: Bogoliubov and Pontecorvo do not commute!!



As a result, flavor vacuum gets a non-trivial term:

$$|0\rangle_{e,\mu} \equiv G_\theta^{-1}|0\rangle_{1,2} = |0\rangle_{1,2} + \left[B(m_1, m_2)\ ,\ R^{-1}(\theta)\right]\ |\widetilde{0}\rangle_{1,2}$$

Flavor Vacuum and Condensate Structure

The flavor vacuum is characterized by a condensate structure:

$$|0\rangle_{e,\mu} = \prod_{\mathbf{k}} \prod_{r} \Big[\left(1 - \sin^2\theta \, |V_{\mathbf{k}}|^2\right) - \epsilon^r \sin\theta \, \cos\theta \, |V_{\mathbf{k}}| \left(\alpha^{r\dagger}_{\mathbf{k},1} \beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2} \beta^{r\dagger}_{-\mathbf{k},1} \right)$$

$$+ \epsilon^r \sin^2\theta \, |V_{\mathbf{k}}||U_{\mathbf{k}}| \left(\alpha^{r\dagger}_{\mathbf{k},1} \beta^{r\dagger}_{-\mathbf{k},1} - \alpha^{r\dagger}_{\mathbf{k},2} \beta^{r\dagger}_{-\mathbf{k},2} \right) + \sin^2\theta \, |V_{\mathbf{k}}|^2 \alpha^{r\dagger}_{\mathbf{k},1} \beta^{r\dagger}_{-\mathbf{k},2} \alpha^{r\dagger}_{\mathbf{k},2} \beta^{r\dagger}_{-\mathbf{k},1} \Big] |0\rangle_{1,2}$$

- SU(2) (Perelomov) coherent state.
- This vacuum structure can be dynamically generated in an effective model within a string inspired framework[13].
- This structure necessarily emerges in chiral symmetric models, when mixing is dynamically generated[14]

[13]N.E. Mavromatos, S. Sarkar and W. Tarantino, Phys. Rev. D **80**, 084046 (2009)

[14]M.Blasone, P. Jizba, N.E. Mavromatos and L.S., Phys. Rev. D. **100**, 045027 (2019).

Flavor eigenstates

Defining

$$\alpha^r_{\mathbf{k},\sigma}(t) \;\equiv\; G^{-1}_\theta(t)\,\alpha^r_{\mathbf{k},j}\,G_\theta(t) \qquad\qquad (\sigma,j)=(e,1),(\mu,2)$$

one can construct flavor eigenstates[15]

$$|\nu^r_{\mathbf{k},\sigma}\rangle \;\equiv\; \alpha^{r\dagger}_{\mathbf{k},\sigma}(0)|0\rangle_{e,\mu}$$

In fact

$$Q_{\nu_\sigma}(0)|\nu^r_{\mathbf{k},\sigma}\rangle \;=\; |\nu^r_{\mathbf{k},\sigma}\rangle$$

[15] M. Blasone and G. Vitiello, Phys Rev D **60**, 111302 (1999)

Oscillation formula

Taking

$$\mathcal{Q}_{\sigma\to\rho}(t) \;=\; \langle \nu^r_{\mathbf{k},\sigma}|Q_{\nu_\rho}(t)|\nu^r_{\mathbf{k},\sigma}\rangle\,, \quad \sigma \neq \rho$$

Explicitly:

$$\mathcal{Q}_{\sigma\to\rho}(t) \;=\; \sin^2(2\theta)\left[|U_{\mathbf{k}}|^2 \sin^2\left(\omega^-_{\mathbf{k}} t\right) + |V_{\mathbf{k}}|^2 \sin^2\left(\omega^+_{\mathbf{k}} t\right)\right]$$

$|U_{\mathbf{k}}|^2 = 1 - |V_{\mathbf{k}}|^2$, $\omega^-_{\mathbf{k}} \equiv (\omega_{\mathbf{k},1} - \omega_{\mathbf{k},2})/2$ and $\omega^+_{\mathbf{k}} \equiv (\omega_{\mathbf{k},1} + \omega_{\mathbf{k},2})/2$. This is the QFT oscillation formula[16]. When $m_i/|\mathbf{k}| \to 0$:

$$\mathcal{Q}_{\sigma\to\rho}(t) \;\approx\; \sin^2(2\theta)\sin^2\left(\frac{\pi L}{L_{osc}}\right)$$

with $L_{osc} = 4\pi|\mathbf{k}|/\delta m^2$, which is the standard oscillation formula.

[16] M. Blasone, P.A. Henning and G. Vitiello, Phys. Lett. B **451**, 140 (1999)

Invariance of oscillation formula

Covariant form of flavor oscillation formula[17]:

$$\mathcal{J}^{\mu}_{\sigma\to\rho}(x-y) \;=\; \langle \nu_\sigma(y)|J^{\mu}_{\nu_\rho}(x)|\nu_\sigma(y)\rangle$$

where $|\nu_\sigma(y)\rangle$ is wavepacket state and $J^{\mu}_{\nu_\rho}(x) \equiv \overline{\nu}_\rho(x)\gamma^\mu\nu_\rho(x)$.

It has been proved (in the boson case) that[18]

- Poincaré is spontaneously broken on flavor vacuum down to $E(3)$
- Oscillation formula is Lorentz invariant

[17] M. Blasone, P. Pires Pacheco and H. Wan Chan Tseung, Phys. Rev. D **67**, 073011 (2003).
[18] M. Blasone, P. Jizba, N.E. Mavromatos and L. S., Phys. Rev. D **102**, 025021 (2020).

FEUR for neutrino oscillations

Flavor-Energy uncertainty (1)

The flavor-charges are not conserved, i.e. $[Q_{\nu_\sigma}(t)\,,\, H] \neq 0$. It follows a flavor-energy uncertainty relation[19]:

$$\langle \Delta H \rangle \, \langle \Delta Q_{\nu_\sigma}(t) \rangle \;\geq\; \frac{1}{2} \left| \frac{\mathrm{d}\langle Q_{\nu_\sigma}(t) \rangle}{\mathrm{d}t} \right|$$

taking the state $|\psi\rangle \;=\; |\nu^r_{\mathbf{k},\sigma}\rangle$:

$$\Delta Q_{\nu_\sigma}(t) \;=\; \sqrt{\mathcal{Q}_{\sigma\to\sigma}(t)(1 - \mathcal{Q}_{\sigma\to\sigma}(t))}$$

we get

$$\left| \frac{\mathrm{d}\mathcal{Q}_{\sigma\to\sigma}(t)}{\mathrm{d}t} \right| \leq 2\Delta E \sqrt{\mathcal{Q}_{\sigma\to\sigma}(t)(1 - \mathcal{Q}_{\sigma\to\sigma}(t))}$$

[19] M. Blasone, P. Jizba and L.S., Phys. Rev. D **99**, 016014 (2019)

Flavor-Energy uncertainty (2)

The r.h.s. has a maximum when $\mathcal{Q}_{\sigma\to\sigma}(T_h) = 1/2$. Then:

$$\left|\frac{\mathrm{d}\mathcal{Q}_{\sigma\to\sigma}(t)}{\mathrm{d}t}\right| \leq \Delta E$$

By using the triangular inequality and integrating:

$$\Delta E\, T \geq \mathcal{Q}_{\sigma\to\rho}(T)\,, \quad \sigma \neq \rho$$

When $T = T_h$:

$$\Delta E\, T_h \geq \frac{1}{2}$$

which is time-energy uncertainty relation[20].

[20] L. Mandelstam and I.G. Tamm, J. Phys. USSR **9**, 249 (1945)

Neutrino oscillation condition

When $m_i/|\mathbf{k}| \to 0$:

$$\Delta E \; \geq \; \frac{2\,\sin^2 2\theta}{L_{osc}}$$

This relation is usually interpreted as neutrino oscillation condition[21].

The situation is the same of unstable particles:

$$\Delta E \; \approx \; \frac{1}{2\tau}$$

where the τ is the particle life-time.

As for unstable particles only energy distribution are meaningful. The width of the distribution is related to the oscillation length.

[21] S.M Bilenky, F. von Feilitzsch and W. Potzel, J. Phys. G **35**, 095003 (2008)

Beyond the ultra-relativistic limit

Corrections beyond ultra-relativistic limit:

$$\Delta E \ \geq \ \frac{2\,\sin^2 2\theta}{L_{osc}} \left[1 - \varepsilon(\mathbf{k})\,\cos^2\left(\frac{|\mathbf{k}|L_{osc}}{2}\right)\right]$$

with $\varepsilon(\mathbf{k}) \equiv (m_1 - m_2)^2/(4|\mathbf{k}|^2)$. When $|\mathbf{k}| = \tilde{k} = \sqrt{m_1 m_2}$:

$$\Delta E \ \geq \ \frac{2\,\sin^2 2\theta}{\tilde{L}_{osc}} \ (1 - \chi)$$

where

$$\chi \ = \ \xi\,\sin\left(\frac{\tilde{\omega}_1\tilde{L}_{osc}}{4}\right)\sin\left(\frac{\tilde{\omega}_2\tilde{L}_{osc}}{4}\right) + \cos\left(\frac{\tilde{\omega}_1\tilde{L}_{osc}}{4}\right)\cos\left(\frac{\tilde{\omega}_2\tilde{L}_{osc}}{4}\right) .$$

and $\xi = 2\sqrt{m_1 m_2}/(m_1 + m_2)$.

Conclusions and Perspectives

Conclusions

- Problem: to define flavor states (in presence of mixing), in QFT language
- The study of mixing transformation reveals that flavor and mass representations are unitarily inequivalent
- In order to preserve conservation of the flavor charge in the vertex, at tree level, we have to work in the flavor basis
- Flavor Fock space approach permits to construct flavor eigenstates
- Oscillation formula must present a fast oscillating term. Such formula is the same for every inertial observer
- For flavor states only energy (mass) distributions are meaningful. The width of the distributions depends on the oscillation length.

Perspectives

- Lorentz invariance for neutrino oscillations (explicit proof)
- Dynamical generation of flavor vacuum in phenomenological models
- Functional Integral formulation
- More on quantum information properties related to neutrino oscillations[22]
- Cosmological applications (e.g. CNB), Lorentz violating effects

[22] M. Blasone, F. Illuminati, L. Petruzziello and L.S., in preparation.

Thank you for the attention!

Neutrino wavepacket

Consider first-quantized Dirac equation

$$\left(i\gamma^\mu\,\partial_\mu \otimes \mathbb{I}_2 - \mathbb{I}_4 \otimes M_\nu\right)\,\Psi(x) \;=\; 0$$

Mass-neutrino wavepackets (along z-axis):

$$\left(i\gamma^0\partial_0 + i\gamma^3\partial_3 - m_j\right)\,\psi_j(z,t) \;=\; 0\,, \qquad j = 1, 2$$

Neutrino wavepacket:

$$\Psi(z,t) = \cos\theta\,\psi_1(z,t)\,\otimes\,\nu_1 \,+\, \sin\theta\,\psi_2(z,t)\,\otimes\,\nu_2$$

$$= \left[\psi_1(z,t)\,\cos^2\theta \,+\, \psi_2(z,t)\,\sin^2\theta\right]\,\otimes\,\nu_\sigma + \sin\theta\,\cos\theta\,\left[\psi_1(z,t) - \psi_2(z,t)\right]\,\otimes\,\nu_\rho$$

$$\equiv \psi_\sigma(z,t)\,\otimes\,\nu_\sigma \,+\, \psi_\rho(z,t)\,\otimes\,\nu_\rho$$

ν_1, ν_2 are the eigenstates of M_ν. ν_σ, ν_ρ are flavor eigenstates.

Oscillation probability(1)

Neutrino is produced as a flavor eigenstate if
$\psi_1(z,0) = \psi_2(z,0) = \psi_\sigma(z,0)$, with $\sigma = e, \mu$. Oscillation probability[23]:

$$P_{\nu_\sigma \to \nu_\rho} \;=\; \int_{-\infty}^{+\infty} \mathrm{d}z\, \psi_\rho^\dagger(z,t)\, \psi_\rho(z,t)$$

Explicitly

$$P_{\nu_\sigma \to \nu_\rho} \;=\; \frac{\sin^2 2\theta}{2}\, \left[1 - I_{12}(t)\right]$$

where

$$I_{12}(t) \;=\; \Re e \left[\int_{-\infty}^{+\infty} \mathrm{d}z\, \psi_1^\dagger(z,t)\, \psi_2(z,t)\right]$$

[23] A. E. Bernardini and S. De Leo, Eur. Phys. J. C **37**, 471-480 (2004).

Oscillation probability(2)

Mass neutrino expansion:

$$\psi_j(x) = \sum_r \int_{-\infty}^{+\infty} \frac{\mathrm{d}p_z}{2\pi} \left[u^r_{p_z,j}\, \alpha^r_{p_z,j}\, e^{-i\,\omega_{p_z,j}\,t} + v^r_{-p_z,j} \beta^{r*}_{-p_z,j}\, e^{i\,\omega_{p_z,j}\,t}\right] e^{i\,p_z\,z}\,, \quad .$$

Initial condition:

$$\alpha^r_{p_z,j} = \varphi_\sigma(p_z - p_0)\, u^{r\dagger}_{p_z,j}\, w\,, \quad \beta^{r*}_{-p_z,j} = \varphi_\sigma(p_z - p_0)\, v^{r\dagger}_{-p_z,j}\, w$$

$\varphi_\sigma(p_z - p_0)$ is flavor neutrino distribution at $t = 0$, p_0 is the mean momentum of wavepackets and w is a constant unit-spinor. Then

$$I_{12}(t) \;=\; \int_{-\infty}^{+\infty} \frac{\mathrm{d}p_z}{2\pi}\, \varphi^2_\sigma(p_z - p_0)\, \left(|U_{p_z}|^2 \cos(\omega^-_{p_z} t) + |V_{p_z}|^2 \cos(\omega^+_{p_z} t)\right)$$

For plane-waves the same as QFT formula!

(Weak field) Schwarzschild metric

Schwarzschild metric in weak field approximation and Fermi coordinates:

$$\mathrm{d}s^2 = (1 + 2\phi)\,\mathrm{d}t^2 - (1 - 2\phi)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right)$$

where the gravitational potential

$$\phi(r) = -\frac{GM}{r} \equiv -\frac{GM}{\sqrt{x^2 + y^2 + z^2}}$$

with G being the Newton constant and M the mass of the source.
The non-vanishing tetrad components are:

$$e^0_{\hat{0}} = 1 - \phi\,, \quad e^i_{\hat{j}} = (1 + \phi)\,\delta^i_j$$

Neutrino oscillations in Schwarzschild metric

For a neutrino moving along the x-axis:

$$P_{\sigma\to\rho}(L_P) = \sin^2(2\theta)\,\sin^2\left(\frac{\pi L_p}{L^{osc}}\right)$$

where the proper length is

$$L_p \;=\; x - x_0 + GM\ln\left(\frac{x}{x_0}\right)$$

and the oscillation length is now[24]:

$$L^{osc} \equiv \frac{4\pi E_\ell}{\Delta m^2}\left[1+\phi+\frac{GM}{x-x_0}\ln\left(\frac{x}{x_0}\right)\right]$$

[24] N. Fornengo, C. Giunti, C. W. Kim and J. Song, Phys. Rev. D **56**, 1895 (1997).

TEUR in Schwarzschild metric

The local energy is now:

$$E_\ell = (1-\phi)\, E$$

We can write down the TEUR[25]:

$$\Delta E_\ell \geq \frac{2\sin^2(2\theta)}{L_{eff}^{osc}(M)}$$

where

$$L_{eff}^{osc}(M) \equiv \frac{4\pi E_\ell}{\Delta m^2}\left[1+2\phi+\frac{2GM}{x-x_0}\ln\left(\frac{x}{x_0}\right)\right]$$

The neutrino "*lifetime*" is now longer.

[25]M. Blasone, G. Lambiase, G.G. Luciano, L. Petruzziello and L. S., Class. Quant. Grav. **37**, 155004 (2020).