Fractional equations of anomalous diffusion - continued

Tadeusz Kosztołowicz

Institute of Physics, Jan Kochanowski University
Uniwersytecka 7, 25-406 Kielce
tadeusz.kosztolowicz@ujk.edu.pl

IF UJK, Kielce, 24/01/2024

Outline

1. Anomalous diffusion equations with fractional derivatives
2. Subdiffusion–immobilization equation

Recipe for creating a fractional derivative

Define an operator $D^{\alpha} \equiv \frac{d^{\alpha}}{dt^{\alpha}} : f(t) \rightarrow g(t)$ that depends on a continuous parameter α. When for $\alpha = n \in \mathcal{N}$ the operator takes the form of an expression derived for an "ordinary" derivative of the order n, D^{α} can be treated as a fractional derivative.

A simple way to find the fractional derivative of a positive order $\alpha \in \mathcal{R}$

- take an equation that has a derivative of natural order n ,
- replace $n \rightarrow \alpha$, $\alpha > 0$,
- in particular, replace $n! = \Gamma(1+n) \rightarrow \Gamma(1+\alpha)$.

The fractional derivative of the order $\alpha < 0$ is called a fractional integral. The method of determining it is slightly different than the one shown above.

The most commonly used fractional derivatives in physics

Caputo fractional derivative, $0 < \alpha < 1$

$$\frac{{}^{C}d^{\alpha}f(t)}{dt^{\alpha}} = \frac{1}{\Gamma(1-\alpha)}\int_0^t [t-u]^{-\alpha} f'(u)du, \quad \mathcal{L}\left[\frac{{}^{C}d^{\alpha}f(t)}{dt^{\alpha}}\right](s) = s^{\alpha}\hat{f}(s) - s^{\alpha-1}f(0)$$

Riemann–Liouville fractional derivative, $0 < \alpha < 1$

$$\frac{{}^{RM}d^{\alpha}f(t)}{dt^{\alpha}} = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t [t-u]^{-1-\alpha} f(u)du, \quad \mathcal{L}\left[\frac{{}^{RL}d^{\alpha}f(t)}{dt^{\alpha}}\right](s) = s^{\alpha}\hat{f}(s)$$

Riesz–Weyl fractional derivative, $1 < \gamma < 2$

$$\frac{d^{\gamma}f(x)}{dx^{\gamma}} = \frac{1}{\Gamma(1-\gamma)}\int_{-\infty}^{x} [x-u]^{-\gamma}\frac{df(u)}{dt}du, \quad \mathcal{F}\left[\frac{d^{\gamma}f(x)}{d|x|^{\gamma}}\right](s) = -|k|^{\gamma}\tilde{f}(k)$$

Application of differential equations with fractional derivatives in physics

Two main ways to get a fractional equation

1. Equation with fractional derivative has a physical basis (the equation is derived from a physical model - an example: anomalous diffusion)
2. Fractional equation is obtained by simply replacing the derivative of natural order with a fractional order derivative. One gets a "more general" equation, but its physical interpretation is rather unknown.

Fractional Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = C(-\hbar^2\Delta)^{\alpha/2}\psi$$

with Riesz spatial fractional derivative, $\alpha \in (0,2)$, N. Laskin, Phys. Rev. E **66**, 056108 (2002), X. Guo, M. Xum J. Math. Phys. **47**, 082104 (2006) , *application to describe fractional dark energy:* R.G. Landim, Phys. Rev. D **104**, 103508 (2021) .

$$(i\hbar)^\alpha\frac{\partial^\alpha\psi}{\partial t^\alpha} = \hat{H}\psi$$

with Caputo time fractional derivative, $\alpha \in (0,1)$, M. Naber, J. Math. Phys. **45**, 3339 (2004), A. Iomin, Phys. Rev. E **80**, 022103 (2009) .

Other generalized physical equations with fractional derivatives (we do not consider here if there is a physical basis for these equations)

- fractional Dirac equation, S. Muslih et al., J. Phys. A **43**, 055203 (2010), A. Raspini, Phys. Scripta **64**, 20 (2001)
- fractional Schrödinger-Klein-Gordon equation, J. Blackledge et al., Math. Aeterna **3**, 601 (2013)
- fractional Maxwell equations, E.K. Jaradat et al., J. Math. Phys. **53**, 033505 (2012)
- fractional Liouville equation, V.E. Tarasov, Phys. Plasmas **20**, 102101 (2013)
- fractional Newtonian mechanics, D. Baleanu et al., Cent. Europ. J. Phys. **8**, 120 (2010), W.S. Chung, J. Comput. Appl. Math. **290**, 150 (2015), G.U. Varieschi, J. Appl. Math. Phys. **6**, 1247 (2018)
- fractional telegrapher's equation, J. Masoliver, Phys. Rev. E **93**, 052107 (2016)
- fractional oscillator equation, Y.E. Ryabov et al., Phys. Rev. B **66**, 184201 (2002)
- fractional Maxwell model of viscoelastic oscillator, Z.-L. Li et al., J. Vibr. Engin. Techn. **6:5**, (2018)
- . . . and many others

Normal and anomalous diffusion

Normal diffusion

$$\frac{\partial P(x,t|x_0)}{\partial t} = D\frac{\partial^2 P(x,t|x_0)}{\partial x^2}$$

$$P(x,0|x_0) = \delta(x-x_0), \quad P(\pm\infty,t|x_0) = 0$$

$$P(x,t|x_0) = \frac{1}{2\sqrt{\pi D t}} e^{-\frac{(x-x_0)^2}{4Dt}}$$

$$\sigma^2(t) \equiv \int_{-\infty}^{\infty} (x-x_0)^2 P(x,t|x_0)\,dx = 2Dt$$

Normal and anomalous diffusion

Anomalous diffusion

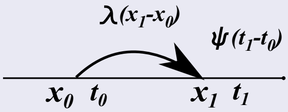
Anomalous diffusion is a process that is not normal diffusion. Important functions that characterize the anomalous diffusion process are different from those of normal diffusion.

$$\sigma^2(t) \sim \begin{cases} t^\beta, \ \beta > 1, \ \text{for superdiffusion}, \\ t, \ \text{for normal diffusion}, \\ t^\alpha, \ 0 < \alpha < 1, \ \text{for subdiffusion}, \\ f(\log t), \ \text{for slow subdiffusion} \end{cases}$$

Qualitatively different diffusion processes

- **Superdiffusion** *(facilitated diffusion) occurs in media where rapid movement of molecules over long distances is common*. Examples: diffusion in turbulent media, epidemic spread (transmission of viruses by air travel), movement of endogeneous intracellular particles in some pathogens, of soil amebas on plastic or glass surfaces in liquid media, mussels movement, cell migration in some biological processes, diffusion in random velocity fields.
- **Normal diffusion** Examples: the most known cases of diffusion, eg diffusion of various substances in water.
- **Ordinary subdiffusion** *occurs in media in which the movement of particles is very hindered*. Examples: transport of some molecules in viscoelastic chromatin network, porous media, living cells, transport of sugars in agarose gel, transport of antibiotic in bacterial biofilm.
- **Slow subdiffusion (ultraslow diffusion)** *(hindered subdiffusion)* Examples: transport of water in aqueous sucrose glasses, language dynamics, diffusion in very crowded media.

Normal and anomalous diffusion



$\langle \lambda^2(x) \rangle < \infty$ $\langle \psi(t) \rangle < \infty$	$\langle \lambda^2(x) \rangle < \infty$ $\langle \psi(t) \rangle = \infty$	$\langle \lambda^2(x) \rangle = \infty$ $\langle \psi(t) \rangle < \infty$
normal diffusion	ordinary subdiffusion	superdiffusion
$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial x^2}$ $\lambda(x) \sim e^{-x^2/2\sigma}$ $\psi(t) \sim e^{-t/\tau}$	$\frac{\partial C}{\partial t} = D_\alpha \frac{\partial_{RL}^{1-\alpha}}{\partial t^{1-\alpha}} \frac{\partial^2 C}{\partial x^2}$ $\lambda(x) \sim e^{-x^2/2\sigma}$ $\psi(t) \sim \left(\frac{\tau}{t}\right)^{1+\alpha}, \; t \gg \tau, \; 0 < \alpha < 1$	$\frac{\partial C}{\partial t} = D_\beta \frac{\partial^\beta C}{\partial x^\beta}$ $\lambda(x) \sim \sigma^{-\beta} \, \lvert x \rvert^{-1-\beta}$ $\lvert x \rvert \gg \sigma, \; 1 < \beta < 2$ $\psi(t) \sim e^{-t/\tau}$

Slow subdiffusion

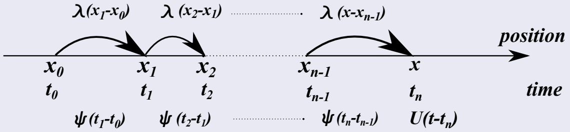
$$\frac{\partial C(x,t)}{\partial t} = \int_0^t K(\log(t')) \frac{\partial^2 C(x, t-t')}{\partial x^2}$$

slow sub. $\langle t^\rho \rangle = \infty$ for $\rho > 0$

ordinary sub. $\langle t^\rho \rangle = \infty$ for $\rho > \alpha$

Continuous time random walk (CTRW) model

E.W. Montroll, G.H. Weiss, J. Math. Phys. **6**, 167 (1965), A. Compte, Phys. Rev. E **53**, 4191 (1996), R. Metzler, J. Klafter, Phys. Rep. **339**, 1 (2000), I.M. Sokolov, J. Klafter, Chaos **15**, 026103 (2005), E. Barkai et al., Phys. Rev. E **61**, 132 (2000) and many others



$$P(x,t|x_0) = \sum_{n=0}^{\infty} Q_n(t) P_n(x|x_0)$$

$Q_n(t)$ is a probability that a diffusing particle takes n step in the time interval $(0,t)$, $P_n(x|x_0)$ is a probability density that the particle will be at x after making n jumps, x_0 is the initial particle position, U is the probability that a particle does not change its position after the last jump, $U(t) = 1 - \int_0^t \psi(t')dt'$

$$Q_n(t) = (\underbrace{\psi *_t \psi *_t \ldots *_t \psi}_{n \text{ times}} *_t U)(t), \quad P_n(x|x_0) = (\underbrace{\lambda *_x \lambda *_x \ldots *_x \lambda}_{n \text{ times}})(x),$$

$$P(x,t|x_0) = \sum_{n=0}^{\infty} Q_n(t) P_n(x|x_0)$$

$Q_n(t)$ is a probability that a diffusing particle takes n step in the time interval $(0,t)$, $P_n(x|x_0)$ is a probability density that the particle will be at x after making n jumps, x_0 is the initial particle position, U is the probability that a particle does not change its position after the last jump, $U(t) = 1 - \int_0^t \psi(t')dt'$

$$Q_n(t) = (\underbrace{\psi *_t \psi *_t \ldots *_t \psi}_{n \ times} *_t U)(t), \;\; P_n(x|x_0) = (\underbrace{\lambda *_x \lambda *_x \ldots *_x \lambda}_{n \ times})(x),$$

$$(f *_t h)(t) = \int_0^t f(t')h(t-t')dt', \;\; \mathcal{L}[(f *_t h)(t)](s) = \mathcal{L}[f(t)](s)\mathcal{L}[h(t)](s) \equiv \hat{f}(s)\hat{g}(s)$$

$$\hat{U}(s) = \mathcal{L}[1 - \int_0^t \psi(t')dt'] = \frac{1-\hat{\psi}(s)}{s}$$

$$(f *_x h)(x) = \int_{-\infty}^{\infty} f(x')h(x-x')dx', \;\; \mathcal{F}[(f *_t h)(x)](k) = \mathcal{F}[f(x)](k)\mathcal{F}[h(x)](k) \equiv \tilde{f}(k)\tilde{g}(k)$$

$$\hat{\tilde{P}}(k,s) = \frac{1-\hat{\psi}(s)}{s} \sum_{n=0}^{\infty} [\hat{\psi}(s)\tilde{\lambda}(k)]^n = \frac{1-\hat{\psi}(s)}{s} \frac{1}{[1-\hat{\psi}(s)\tilde{\lambda}(k)]}$$

$\hat{\psi}(s) = \int_0^\infty \exp(-st)\psi(t)dt$, $\tilde{\lambda}(k) = \int_{-\infty}^\infty \exp(ikx)\lambda(x)dx$

$\exp(u) = \sum_{j=0}^\infty u^j/j!$,

$\left\langle t^j \right\rangle = \int_0^\infty t^j \psi(t)dt$, $\left\langle x^j \right\rangle = \int_{-\infty}^\infty x^j \lambda(x)dx$,

$$\tilde{\lambda}(k) = \sum_{j=0}^{\infty} (ik)^j \frac{\left\langle x^j \right\rangle}{j!} \ , \ \hat{\psi}(s) = \sum_{j=0}^{\infty} (-s)^j \frac{\left\langle t^j \right\rangle}{j!} \ .$$

$$\langle t \rangle = - \left. \frac{d\hat{\psi}(s)}{ds} \right|_{s=0} , \ \left\langle x^2 \right\rangle = - \left. \frac{d^2\tilde{\lambda}(|k|)}{d|k|^2} \right|_{k=0}$$

$$\hat{\psi}(s) = 1 - \tau s + \tau^2 s^2/2 - \ldots,$$
$$\langle t \rangle = -\left.\frac{d\hat{\psi}(s)}{ds}\right|_{s=0} = \tau$$
$$\tilde{\lambda}(k) = 1 - \rho^2 |k|^2/2 + \ldots,$$
$$\left\langle x^2 \right\rangle = -\left.\frac{d^2\tilde{\lambda}(k)}{dk^2}\right|_{k=0} = \rho^2$$

$$\hat{\psi}(s) = 1 - \tau s^\alpha + \tau^2 s^{2\alpha}/2 - \ldots,$$
$$\langle t \rangle = -\left.\frac{d\hat{\psi}(s)}{ds}\right|_{s=0} = \left.\frac{\tau}{s^{1-\alpha}}\right|_{s=0} = \infty,\ 0 < \alpha < 1$$
$$\tilde{\lambda}(|k|) = 1 - \rho^2 |k|^\gamma/2 + \ldots,$$
$$\left\langle x^2 \right\rangle = -\left.\frac{d^2\tilde{\lambda}(|k|)}{d|k|^2}\right|_{k=0} = \left.\frac{\rho^2\gamma(\gamma-1)}{|k|^{2-\gamma}}\right|_{k=0} = \infty,\ 1 < \gamma < 2$$

$$\hat{\tilde{P}}(k,s) = \frac{1-\hat{\psi}(s)}{s}\frac{1}{[1-\hat{\psi}(s)\tilde{\lambda}(k)]}$$

$$\hat{\psi}(s) = 1-\tau s^{\alpha},\ s\to 0;\ \ \tilde{\lambda}(|k|) = 1-\rho^2|k|^2/2,\ |k|\to 0,\ D_\alpha = \rho^2/2\tau$$

$$s\hat{\tilde{P}}(k,s) - P(x,0) = -D_\alpha s^{1-\alpha}|k|^2\hat{\tilde{P}}(k,s),$$

$$\frac{\partial P(x,t)}{\partial t} = D_\alpha \frac{{}^{RL}\partial^{1-\alpha}}{\partial t^{1-\alpha}}\frac{\partial^2 P(x,t)}{\partial x^2}$$

$$\hat{\psi}(s) = 1-\tau s,\ s\to 0;\ \ \tilde{\lambda}(|k|) = 1-\rho^\gamma|k|^\gamma/2,\ |k|\to 0,\ D_\gamma = \rho^\gamma/2\tau$$

$$s\hat{\tilde{P}}(k,s) - P(x,0) = -D_\alpha|k|^\gamma\hat{\tilde{P}}(k,s),$$

$$\frac{\partial P(x,t)}{\partial t} = D_\gamma\frac{\partial^\gamma P(x,t)}{\partial x^\gamma}$$

T. Kosztołowicz, J. Phys. A: Math. Gen. **37**, 10779 (2004)

$$\mathcal{L}^{-1}\left[s^{\nu}\mathrm{e}^{-as^{\beta}}\right](t) = f_{\nu,\beta}(t;a) = \frac{1}{t^{1+\nu}}\sum_{j=0}^{\infty}\frac{1}{j!\Gamma(-\nu-\beta j)}\left(-\frac{a}{t^{\beta}}\right)^{j}$$

$$a, \beta > 0,$$

The function $f_{\nu,\beta}$ is a special case of the H-Fox function and the Wright function.

Subdiffusion–immobilization process

T. Kosztołowicz, *Subdiffusion with particle immobilization process described by differential equation with Riemann–Liouville type fractional time derivative*, Phys. Rev. E **108**, 014132 (2023)

The diffusion–immobilization process has been observed in diffusion of chemically reactive gases in polymer layer systems, transport of molecules in zeolites, signal transduction in living cells, drug release processes, immobilization of enzymes affecting catalytic reactions, oxygen diffusion trough gels, diffusion and immobilization of dyes and lithium ions in some nanocomposite anodes. Both processes mentioned above can occur in diffusion of antibiotic in a bacterial biofilm. One of bacteria defence mechanisms is to disintegrate the antibiotic molecules, the process can be described by diffusion-reaction equations. In the other one bacteria can thicken the biofilm immobilizing antibiotic molecules. The immobilized molecules have not disappeared, they can further interact with the environment. The process of immobilization of diffusing molecules has been described by a diffusion equation with an additional term describing the immobilization process.

(Sub)diffusion–reaction versus (sub)diffusion–immobilization

Subdiffusion–reaction

$$\int_{-\infty}^{\infty} P(x,t)dx < 1, \int_0^{\infty} \psi(t)dt = 1 \qquad (1)$$

$$\hat{\psi}(s) = \frac{1}{1+\tau s^{\alpha}}, \int_0^{\infty} \psi(t)dt \equiv \hat{\psi}(0) = 1 \qquad (2)$$

Subdiffusion–immobilization

$$\int_{-\infty}^{\infty} P(x,t)dx = 1, \int_0^{\infty} \psi(t)dt < 1 \qquad (3)$$

$$\hat{\psi}(s) = \frac{1}{1+\tau\gamma+\tau s^{\alpha}}, \int_0^{\infty} \psi(t)dt \equiv \hat{\psi}(0) = \frac{1}{1+\tau\gamma} \qquad (4)$$

The probability p_s of stopping the molecule permanently is $p_s = 1 - \hat{\psi}(0) = \tau\gamma/(1 + \tau\gamma)$, then $\gamma = p_s/[(1 - p_s)\tau]$. The function ψ is interpreted here as a probability density of waiting time for a particle to jump provided that the particle has not been permanently immobilized by this time.

$$s\hat{P}(x,s) - P(x,0) = D\frac{s^{1-\alpha}}{1+\gamma s^{-\alpha}}\frac{\partial^2 \hat{P}(x,s)}{\partial x^2}. \tag{5}$$

$$\mathcal{L}^{-1}\left[\frac{s^{1-\alpha}}{1+\gamma s^{-\alpha}}\hat{f}(s)\right](t) = \frac{{}^{RL}_{F}d^{1-\alpha}f(t)}{dt^{1-\alpha}}, \tag{6}$$

where

$$\frac{{}^{RL}_{F}d^{1-\alpha}f(t)}{dt^{1-\alpha}} = \frac{d}{dt}\int_0^t F_\alpha(t-t';\gamma)f(t')dt' \tag{7}$$

is the Riemann–Liouville type fractional derivative with the kernel F_α which is defined by its Laplace transform

$$\hat{F}_\alpha(s;\gamma) = \frac{1}{\gamma + s^\alpha}. \tag{8}$$

Calculation of the inverse transform of Eq. (8) is usually done by power series expansion of the function when $\gamma/s^{\alpha} < 1$, and then inverting the transform term by term using the formula $\mathcal{L}^{-1}[1/s^{\beta}](t) = t^{\beta-1}/\Gamma(\beta)$, $\beta > 0$. The result is the Mittag-Leffler function. However, this procedure is valid for relatively large values of the parameter s, which correspond to small values of time variable. To get the inverse Laplace transform over the whole time domain we propose to use the following method:

1. instead of $\hat{F}_\alpha$ Eq. (8) find the inverse transform of $\hat{F}_\alpha(s,\gamma)\mathrm{e}^{-as^\mu}$, $a, \mu > 0$,
2. expand $\hat{F}_\alpha$ in a power series of s considering both cases $s^\alpha > \gamma$ and $s^\alpha < \gamma$ separately,
3. use the formula

$$\mathcal{L}^{-1}\left[s^\nu \mathrm{e}^{-as^\mu}\right](t) \equiv f_{\nu,\mu}(t;a) \tag{9}$$
$$= \frac{1}{t^{\nu+1}} \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(-n\mu-\nu)} \left(-\frac{a}{t^\mu}\right)^n,$$

   $a, \mu > 0$,

4. calculate the limit of $a \to 0^+$ in the obtained functions. We note that

$$f_{\nu,\mu}(t;0^+) = \frac{1}{t^{\nu+1}\Gamma(-\nu)}, \tag{10}$$

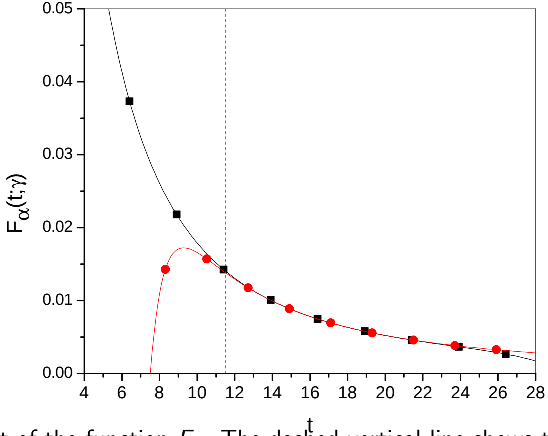
   and the result is independent of the parameter μ.

$$\frac{\mathrm{e}^{-as^{\mu}}}{\gamma + s^{\alpha}} = \begin{cases} \mathrm{e}^{-as^{\mu}} \sum_{n=0}^{\infty} (-\gamma)^n s^{-(n+1)\alpha}, \ s > \gamma^{1/\alpha}, \\ \frac{\mathrm{e}^{-as^{\mu}}}{\gamma} \sum_{n=0}^{\infty} \left(-\frac{1}{\gamma}\right)^n s^{n\alpha}, \ s < \gamma^{1/\alpha}, \end{cases} \tag{11}$$

and Eqs. (9) and (10) we obtain

$$F_{\alpha}(t;\gamma) = \begin{cases} \frac{1}{t^{1-\alpha}} E_{\alpha,\alpha}(-\gamma t^{\alpha}), \ t < t_b, \\ -\frac{1}{\gamma^2 t^{1+\alpha}} \tilde{E}_{\alpha,\alpha}\left(-\frac{1}{\gamma t^{\alpha}}\right), \ t > t_b, \end{cases} \tag{12}$$

where $E_{\alpha,\beta}(u) = \sum_{n=0}^{\infty} \frac{u^n}{\Gamma(\alpha n+\beta)}$, $\alpha, \beta > 0$, is the two–parameter Mittag–Leffler (ML) function, $\tilde{E}_{\alpha,\beta}(u) = \sum_{n=0}^{\infty} \frac{u^n}{\Gamma(-\alpha n-\beta)}$ is a generalization of the ML function for negative parameters.



Plot of the function F_α. The dashed vertical line shows the location of the parameter $t_b = 11.5$. The solid line with squares is the plot of the upper function in Eq. (12) which describes F_α for $t < t_b$, the solid line with circles is the plot of the lower function in Eq. (12) which represents F_α for $t > t_b$. In the numerical calculations, the leading 20 terms in the series appearing in the functions $E_{\alpha,\alpha}$ and $\tilde{E}_{\alpha,\alpha}$ have been included, $\alpha = 0.7$, $\gamma = 0.6$, and $D = 10$.

$$\hat{P}(x,s) = \frac{\sqrt{\gamma + s^{\alpha}}}{2s\sqrt{D}}\, \mathrm{e}^{-|x|\frac{\sqrt{\gamma+s^{\alpha}}}{\sqrt{D}}}. \quad (13)$$

$$P(x,t\to\infty) \equiv P_{st}(x) = \frac{1}{2}\sqrt{\frac{\gamma}{D}}\, \mathrm{e}^{-\sqrt{\frac{\gamma}{D}}|x|}. \quad (14)$$

$$\sigma^2(t) \approx \frac{2D}{\gamma}\left[1 - \frac{1}{\gamma\Gamma(1-\alpha)t^{\alpha}}\right]. \quad (15)$$

$$\sigma^2(t)(t\to\infty) = \frac{2D}{\gamma}$$

When $s^\alpha > \gamma$ we obtain

$$\hat{P}(x,s) = \frac{1}{2\sqrt{D}s^{1-\alpha/2}}\left(1 - \frac{b_1}{s^{\alpha/2}} + \frac{b_2}{s^\alpha}\right)\mathrm{e}^{-\frac{|x|}{\sqrt{D}}s^{\alpha/2}}, \qquad (16)$$

where $b_1 = \gamma|x|/2\sqrt{D}$ and $b_2 = (\gamma/2)(1 + |x|^2\gamma/2\sqrt{D})$. If $s^\alpha < \gamma$, we get

$$\hat{P}(x,s) = \frac{\sqrt{\gamma}}{2s\sqrt{D}}\mathrm{e}^{-\sqrt{\frac{\gamma}{D}}|x|(1+\frac{s^\alpha}{2\gamma})}\left[1 + \frac{s^\alpha}{2\gamma} - b\frac{s^{2\alpha}}{\gamma^2}\right], \qquad (17)$$

where $b = \sqrt{\gamma/D}|x| + 1/8$.

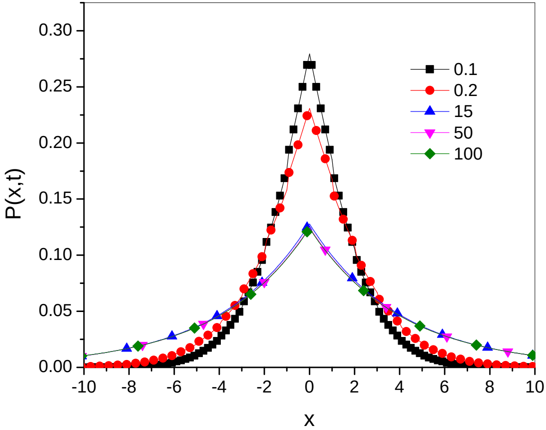
Eqs. (9) and (16) provide the Green's functions in the limit of short time

$$P(x,t) = \frac{1}{2\sqrt{D}}\Big[f_{-1+\alpha/2,\alpha/2}(t;\eta) \tag{18}$$
$$-b_1 f_{-1,\alpha/2}(t;\eta) + b_2 f_{-1-\alpha/2,\alpha,2}(t;\eta)\Big],$$

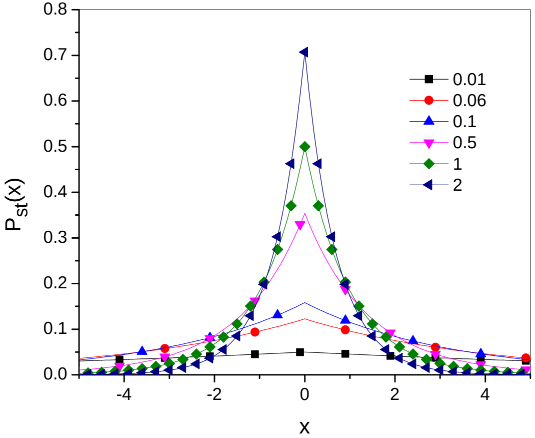
where $\eta = |x|/\sqrt{D}$. From Eqs. (9) and (17) we get the Green's function in the long time limit

$$P(x,t) = \frac{1}{2}\sqrt{\frac{\gamma}{D}}\mathrm{e}^{-\sqrt{\frac{\gamma}{D}}|x|}\Big[f_{-1,\alpha}(t;\xi) \tag{19}$$
$$+\frac{1}{2\gamma}f_{\alpha-1,\alpha}(t;\xi) - \frac{b}{\gamma^2}f_{2\alpha-1,\alpha}(t;\xi)\Big],$$

where $\xi = |x|/2\sqrt{D\gamma}$.



Plots of Green's functions for times given in the legend. The plots represent the function Eq. (16) for $t = 0.1, 0.5$ and Eq. (17) for $t = 15, 50, 100$.



Plots of the function P_{st} Eq. (14) for different values of the ratio γ/D given in the legend.

Final remarks

In equations describing "special" diffusion processes, new fractional derivatives can be involved.

For diffusion equations with new fractional derivatives, new methods for solving fractional differential equations often have to be found.

The search for new fractional equations describing various anomalous diffusion processes is still ongoing ...

Thank you for your attention