Fractional calculus in physics. Anomalous diffusion equations.

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Outline

1. Fractional calculus: derivatives of fractional order
2. Anomalous diffusion equations with fractional derivatives
3. G–subdiffusion equation

Recipe for creating a fractional derivative

Define an operator $D^{\alpha} \equiv \frac{d^{\alpha}}{dt^{\alpha}} : f(t) \rightarrow g(t)$ that depends on a continuous parameter α. When for $\alpha = n \in \mathcal{N}$ the operator takes the form of an expression derived for an "ordinary" derivative of the order n, D^{α} can be treated as a fractional derivative.

A simple way to find the fractional derivative of a positive order $\alpha \in \mathcal{R}$

- take an equation that has a derivative of natural order n ,
- replace $n \rightarrow \alpha$, $\alpha > 0$,
- in particular, replace $n! = \Gamma(1+n) \rightarrow \Gamma(1+\alpha)$.

The fractional derivative of the order $\alpha < 0$ is called a fractional integral. The method of determining it is slightly different than the one shown above.

A long history of fractional calculus

First known attempt: Leibniz's letter to de l'Hospital (1695) - an attempt to define 1/2-order differential

$$d^{1/2}x = x\sqrt{dx : x}$$

Lacroix's fractional derivative (1819)

$$\frac{d^n x^m}{dx^n} = \frac{m!}{(m-n)!}x^{m-n}, \; m, n \in \mathcal{N}, \; m \geqslant n.$$

.

$$\frac{d^\alpha x^m}{dx^\alpha} = \frac{\Gamma(1+m)}{\Gamma(1+m-\alpha)}x^{m-\alpha}, \; \alpha \in \mathcal{R}, \; \alpha > 0.$$

Liouville's fractional derivative (1832)

$$\frac{d^n \exp(ax)}{dx^n} = a^n \exp(ax) \; .$$

$$\frac{d^\alpha \exp(ax)}{dx^\alpha} = a^\alpha \exp(ax) \; .$$

More modern forms of fractional derivatives

Integral-differential operator, "non-local" derivative (wide applications in physics)

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{d^n}{dt^n}\int_a^t K(t-\tau,\alpha)f(\tau)d\tau, \;\; n=[\alpha]+1,$$
$$\frac{d^\alpha f(t)}{dt^\alpha} = \int_a^t K(t-\tau,\alpha)\frac{d^n f(\tau)}{d\tau^n}d\tau,$$

usually $a=0$ or $a=-\infty$; it is also possible to integrate over the interval (a,∞)

The limit of the generalized differential quotient, "conformable", "local" fractional derivative (rather rare applications in physics), e.g.

$$\frac{d^\alpha f(t)}{dt^\alpha} = \lim_{\epsilon\to 0}\frac{f(t\mathrm{e}^{\epsilon t^{-\alpha}})-f(t)}{\epsilon}.$$

Different definitions of fractional derivatives

G.Sales Teodoro, J.A.Tenreiro Machado, E.Capelas de Oliveira, A review of definitions of fractional derivatives and other operators, J. Computational Phys. 388, 195 (2019)

- Grünwald-Letnikov left-sided derivative

$$ {}^{\mathrm{GL}}\mathrm{D}^{\alpha}_{a+}[f(x)] = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{\lfloor n\rfloor}(-1)^k\frac{\Gamma(\alpha+1)f(x-kh)}{\Gamma(k+1)\Gamma(\alpha-k+1)},\ nh = x-a $$

- Grünwald-Letnikov right-sided derivative

$$ {}^{\mathrm{GL}}\mathrm{D}^{\alpha}_{b-}[f(x)] = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{\lfloor n\rfloor}(-1)^k\frac{\Gamma(\alpha+1)f(x+kh)}{\Gamma(k+1)\Gamma(\alpha-k+1)},\ nh = b-x $$

- Riemann-Liouville left-sided derivative

$$ {}^{\mathrm{RL}}\mathrm{D}^{\alpha}_{a+}[f(x)] = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_a^x (x-\xi)^{n-\alpha-1}f(\xi)\,\mathrm{d}\xi,\ x\geq a $$

- Riemann-Liouville right-sided derivative

$$ {}^{\mathrm{RL}}\mathrm{D}^{\alpha}_{b-}[f(x)] = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_x^b (\xi-x)^{n-\alpha-1}f(\xi)\,\mathrm{d}\xi,\ x\leq b $$

- Caputo left-sided derivative

$$ {}^{\mathrm{C}}\mathrm{D}^{\alpha}_{a+}[f(x)] = \frac{1}{\Gamma(n-\alpha)}\int_a^x (x-\xi)^{n-\alpha-1}\frac{\mathrm{d}^n}{\mathrm{d}\xi^n}[f(\xi)]\,\mathrm{d}\xi,\ x\geq a $$

- Caputo right-sided derivative

- Weyl

$$ {}_{x}\mathrm{D}_{\infty}^{\alpha}[f(x)] = \mathrm{D}_{-}^{\alpha}[f(x)] = (-1)^{m}\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^{n}\left[{}_{x}\mathrm{W}_{\infty}^{\alpha}[f(x)]\right] $$

with $\left[{}_{x}\mathrm{W}_{\infty}^{\alpha}[f(x)]\right] = \frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(t-x)^{\alpha-1}f(t)\,\mathrm{d}t.$

- Marchaud

$$ \mathrm{D}_{+}^{\alpha}[f(x)] = \frac{\alpha}{\Gamma(1-\alpha)}\int_{-\infty}^{x}\frac{f(x)-f(\xi)}{(x-\xi)^{1+\alpha}}\mathrm{d}\xi $$

- Hadamard

$$ \mathrm{D}_{+}^{\alpha}[f(x)] = \frac{x}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_{a}^{x}\left(\ln\frac{x}{\tau}\right)^{2-\alpha}f(\tau)\frac{\mathrm{d}\tau}{\tau} $$

- Chen

$$ \mathrm{D}_{c}^{\alpha}[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_{c}^{x}(x-\xi)^{-\alpha}f(\xi)\,\mathrm{d}\xi,\ x \geq c $$

- Davidson-Essex

$$ \mathrm{D}_{0}^{\alpha}[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}^{n+1-k}}{\mathrm{d}x^{n+1-k}}\int_{0}^{x}(x-\xi)^{-\alpha}\frac{\mathrm{d}^{k}}{\mathrm{d}\xi^{k}}[f(\xi)]\,\mathrm{d}\xi $$

- Canavati

$$ {}_a\mathrm{D}_x^{\nu}[f(x)] = \frac{1}{\Gamma(1-\mu)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x (x-\xi)^{\mu}\frac{\mathrm{d}^n}{\mathrm{d}\xi^n}[f(\xi)]\,\mathrm{d}\xi,\ n=\lfloor\nu\rfloor,\ \mu=n- $$

- Jumarie

$$ \mathrm{D}_x^{\alpha}[f(x)] = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_0^x (x-\xi)^{n-\alpha-1}[f(\xi)-f(0)]\,\mathrm{d}\xi $$

- Erdélyi-Kober derivative [78]:

$$ D_{a+;\sigma,\eta}^{\alpha} f(x) = x^{-\sigma(\alpha+\eta)}\left(\frac{1}{\sigma x^{\sigma-1}}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n x^{\sigma(\alpha+n+\eta)} I_{a+;\sigma,\eta}^{\alpha+n} f(x),\ \alpha>-n $$

with

$$ I_{a+;\sigma,\eta}^{\alpha} f(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\int_a^x f(\tau)(x^{\sigma}-\tau^{\sigma})^{\alpha-1}\tau^{\sigma\eta+\sigma-1}\mathrm{d}\tau,\ \alpha>0 $$

and

$$ D_{b^-;\sigma,\eta}^{\alpha} f(x) = x^{\sigma\eta}\left(-\frac{1}{\sigma x^{\sigma-1}}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n x^{\sigma(n-\eta)} I_{b^-;\sigma,\eta-n}^{\alpha+n} f(x),\ \alpha>-n, $$

- Regularized Liouville derivative [72]:

$$ D_f^{\alpha} f(t) = \frac{1}{\Gamma(-\alpha)}\int_0^{\infty}\tau^{-\alpha-1}\left[f(t-\tau)-\sum_{m=0}^{N-1}\frac{(-1)^m f^{(m)}(t)}{m!}\tau^m\right]\mathrm{d}\tau, $$

with $N=\lfloor\alpha\rfloor+1$ and $\lfloor\alpha\rfloor$ the integer part of α.

- Riesz/Feller derivative [72]:

$$ D_{\theta}^{\alpha} f(t) = \frac{1}{2\sin(\alpha\pi)\Gamma(-\alpha)}\int_{\mathbb{R}} f(t-\tau)\sin\left[(\alpha+\theta\cdot\mathrm{sgn}(\tau))\frac{\pi}{2}\right]|\tau|^{-\alpha-1}\mathrm{d}\tau, $$

with $\theta\in\mathbb{R}$ and $\mathrm{sgn}(\cdot)$ denoting the signal function.

- Two-sided derivative [72]:

$$ D_{\mathrm{C}}^{\gamma} f(t) = \lim_{h\to0^+} h^{-\gamma}\sum_{n=-\infty}^{+\infty}(-1)^n\frac{\Gamma(\gamma+1)}{\Gamma\left(\frac{\gamma+\theta}{2}-n+1\right)\Gamma\left(\frac{\gamma-\theta}{2}+n+1\right)} f(t-nh), $$

with $\gamma>-1$.

- Hilfer derivative [34]:

$$ D_{a\pm}^{\alpha,\mu} f(t) = \pm I_{a\pm}^{\mu(1-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) I_{a\pm}^{(1-\mu)(1-\alpha)} f(t),\ 0\le\mu\le1, $$

where $0<\alpha<1$ and

$$ I_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t f(\tau)(t-\tau)^{\alpha-1}\mathrm{d}\tau,\ t\ge a $$

$$ I_{b-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b f(\tau)(\tau-t)^{\alpha-1}\mathrm{d}\tau,\ t\le b. $$

- Hilfer-Katugampola [63]:

$$^{\rho}D_{a\pm}^{\alpha,\beta} f(x) = \left[\pm {}^{\rho}I_{a\pm}^{\beta(1-\alpha)} \left(t^{1-\rho} \frac{\mathrm{d}}{\mathrm{d}t} \right) {}^{\rho}I_{a\pm}^{(1-\beta)(1-\alpha)} \right] f(t), \ \rho > 0,$$

where $0 < \alpha < 1$, $0 \leq \beta \leq 1$ and

$$^{\rho}I_{a+}^{\alpha} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{a}^{x} f(\tau)(x^{\rho} - \tau^{\rho})^{\alpha-1} \mathrm{d}\tau, \ x > a$$

- Kolwankar [46]:

$$D^{\alpha} f(x) = \lim_{x' \to x} D_{x}^{\alpha} [f(x') - f(x)],$$

where D_{x}^{α} is the Riemann-Liouville fractional derivative.

- Chen [23]:

$$\frac{\partial f(x)}{\partial x^{\alpha}} = \lim_{s \to x} \frac{f(x) - f(s)}{x^{\alpha} - s^{\alpha}}.$$

$$T_\alpha f(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}.$$

- Katugampola [40]:

$$D^\alpha f(x) = \lim_{\varepsilon \to 0} \frac{f(xe^{\varepsilon x^{-\alpha}}) - f(x)}{\varepsilon}.$$

- $\mathcal{M}$ [91]:

$$\mathscr{D}_M^{\alpha,\beta} f(x) = \lim_{\varepsilon \to 0} \frac{f(xE_\beta(\varepsilon x^{-\alpha})) - f(x)}{\varepsilon}, \ \beta > 0.$$

- Deformable [104]:

$$\mathcal{D}^\alpha f(x) = \lim_{\varepsilon \to 0} \frac{(1+\varepsilon\beta) f(x+\varepsilon\alpha) - f(x)}{\varepsilon}, \ \alpha+\beta = 1.$$

- Beta [10]:

$${}_0^A D_x^\beta (f(x)) = \lim_{\varepsilon \to 0} \frac{f\left(x+\varepsilon\left(x+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right) - f(x)}{\varepsilon}, \ \beta \in (0,1].$$

- AGO [6]:

$$f^{(\alpha)}(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon(k(x))^{1-\alpha}) - f(x)}{\varepsilon}.$$

- Generalized [3]:

$${}_G D^\alpha f(x) = \lim_{\varepsilon \to 0} \frac{f\left(x - k(x) + k(x) e^{\frac{\varepsilon (k(x))^{-\alpha}}{k'(x)}}\right) - f(x)}{\varepsilon}.$$

- $\mathcal{V}$-truncated [94]:

$${}_i^\rho \mathcal{V}_{\gamma,\beta,\alpha}^{\delta,p,q} f(x) = \lim_{\varepsilon \to 0} \frac{f\left(x \, {}_i H_{\gamma,\beta,p}^{\rho,\delta,q}(\varepsilon x^{-\alpha})\right) - f(x)}{\varepsilon},$$

where $\gamma, \beta, \rho, \delta \in \mathbb{C}$, $p, q > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > 0$, $\mathrm{Re}(\rho) > 0$, $\mathrm{Re}(\delta) >$

$${}_i H_{\gamma,\beta,p}^{\rho,\delta,q}(x) = \Gamma(\beta) \sum_{k=0}^{i} \frac{x^k (\rho)_{kq}}{\Gamma(\gamma k + \beta)(\delta)_{kp}}.$$

- Conformable of β-type in the Riemann-Liouville sense [60]:

$${}^{AR}{}_a^\beta \mathcal{D}_t^\alpha f(x) = {}^A{}_a^n \mathcal{D}_t^\alpha ({}^A{}_a^{n-\beta} \mathcal{I}_t f(x)), \qquad \mathrm{Re}(\beta) > 0, \ n = [\mathrm{Re}(\beta)$$

- General conformable [101]:

$$D_\psi^p f(u) = \lim_{\varepsilon \to 0} \frac{f(u+\varepsilon\psi(u,p)) - f(u)}{\varepsilon}.$$

- Caputo-Fabrizio [19]:

$$\mathscr{D}_t^{(\alpha)} f(t) = \frac{M(\alpha)}{1-\alpha}\int_a^t \dot{f}(\tau)e^{-\frac{\alpha(t-\tau)}{1-\alpha}}\,\mathrm{d}\tau.$$

- Atangana-Baleanu Caputo type [9]:

$${}^{ABC}{}_bD_t^{\alpha}(f(t)) = \frac{B(\alpha)}{1-\alpha}\int_b^t f'(x)E_\alpha\left(-\alpha\frac{(t-x)^\alpha}{1-\alpha}\right)\mathrm{d}x,\ t>b.$$

- Atangana-Baleanu Riemann-Liouville type [9]:

$${}^{ABR}{}_bD_t^{\alpha}(f(t)) = \frac{B(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\int_b^t f(x)E_\alpha\left(-\alpha\frac{(t-x)^\alpha}{1-\alpha}\right)\mathrm{d}x,\ t>b.$$

- Yang et al. [98]:

$${}^{YSM}D_{a+}^{(\alpha)} f(t) = \frac{R(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t f(\tau)e^{-\frac{\alpha(t-\tau)}{1-\alpha}}\,\mathrm{d}\tau,\ t>a.$$

- Generalized Caputo type [86]:

$${}^{gC}D_t^{\alpha,\beta}(f(t)) = \frac{G(\alpha)}{1-\alpha}\int_b^t f'(x)E_\beta\left(-\alpha\frac{(t-x)^\beta}{1-\alpha}\right)\mathrm{d}x,\ \beta\in[0,1]$$

- Generalized Riemann-Liouville type [86]:

$${}^{gRL}D_t^{\alpha,\beta}(f(t)) = \frac{G(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\int_b^t f(x)E_\beta\left(-\alpha\frac{(t-x)^\beta}{1-\alpha}\right)\mathrm{d}x,\ \beta\in[0,$$

- Caputo-Fabrizio with Gaussian kernel [20]:

$${}^{CF}D^{\alpha} f(t) = \frac{1+\alpha^2}{\sqrt{\pi^\alpha(1-\alpha)}}\int_a^t \dot{f}(\tau)e^{-\frac{\alpha(t-\tau)^2}{1-\alpha}}\,\mathrm{d}\tau,\ f(a)=0,\ t>a.$$

- Sun-Hao-Zhang-Baleanu [81]:

$${}^{SE}D_{a+}^{\alpha} f(t) = \frac{M(\alpha)}{(1-\alpha)^{\frac{1}{\alpha}}}\int_a^t f'(\tau)e^{-\frac{\alpha(t-\tau)^\alpha}{1-\alpha}}\,\mathrm{d}\tau,\ t>a.$$

Leibniz rule

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}$$

Classical derivative	Generalized Leibniz rule
Grünwald-Letnikov	$_{GL}D^{\alpha}(fg)(x) = \sum_{i=0}^{\infty} \binom{\alpha}{i} f^{(i)}(x)_{GL}D^{\alpha-i}g(x)$
Riemann-Liouville	$D^{\alpha}(fg)(x) = \sum_{k=0}^{\infty} \binom{\alpha}{k} f^{(k)}(x) D^{\alpha-k}g(x)$
Caputo	${}^{C}D^{\alpha}(fg)(x) = \sum_{k=0}^{\infty} \binom{\alpha}{k}_{*} D^{k}f(x)_{*}D^{\alpha-k}g(x) + g(0)(f(x) - f(0))\frac{t^{-\alpha}}{\Gamma(1-\alpha)}$
Hilfer	$D_{a\pm}^{\alpha,\mu}(fg)(x) = \sum_{m=0}^{\infty} \binom{\alpha}{m} f^{(m)}(x) D_{a\pm}^{\alpha-m,\mu}g(x) + \sum_{k=0}^{\infty} \binom{-(1-\mu)(1-\alpha)}{k} I_{a\pm}^{k+(1-\mu)(1-\alpha)}g(a)(f^{(k)}(x) - f^{(k)}(a))\frac{(x-a)^{-\alpha\mu+\mu-1}}{\Gamma(\mu-\alpha\mu)}$
Weyl	$W_{\pm}^{\alpha}(fg)(x) = \sum_{m=0}^{\infty} \binom{\alpha}{m} f^{(m)}(x) W_{\pm}^{\alpha-m}g(x)$
ψ-Hilfer	${}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}(fg)(x) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \binom{-(1-\beta)(1-\alpha)}{m-l}\binom{\beta(\alpha-1)+1}{l} f^{(m)}(x)^{RL}\mathbb{D}_{a+}^{\alpha-m;\psi}g(x)$ $- \sum_{k=0}^{\infty} \binom{-(1-\beta)(1-\alpha)}{k} I_{a+}^{(1-\beta)(1-\alpha)+k;\psi} g(a) f^{(k)}(a)\frac{[\psi(x)-\psi(a)]^{-1-\beta(\alpha-1)}}{\Gamma(\beta(1-\alpha))}$
Hadamard	$\mathrm{D}_{a+}^{\alpha} fg(x) = \sum_{m=0}^{\infty} \left[\binom{\alpha-1}{m} + \binom{\alpha-1}{m-1}\right] f^{(m)}(x)^{RL}\mathbb{D}_{a+}^{\alpha-m;\psi}g(x).$
Erdélyi-Kober	$D_{a+;\sigma,\eta}^{\alpha} fg(x) = x^{-\sigma(2\eta+\alpha)} \sum_{m=0}^{\infty} \left[\binom{\alpha-1}{m} + \binom{\alpha-1}{m-1}\right] \sum_{k=0}^{m} \frac{\Gamma(-\sigma(\eta+\alpha)+1)}{\Gamma(-\sigma(\eta+\alpha)-k+1)} x^{-k} f^{(m-k)}(x)^{RL}\mathbb{D}_{a+}^{\alpha-m;\psi}g(x)$
ψ-Caputo	${}^{C}\mathbb{D}_{a+}^{\alpha;\psi}(fg)(x) = \sum_{k=0}^{\infty} \binom{\alpha}{k} f^{(k)}(x)^{RL}\mathbb{D}_{a+}^{\alpha-k;\psi}g(x) - \sum_{k=0}^{n-1} \frac{\frac{d^k}{dx^k}[f(x)g(x)](a)}{\Gamma(k-\alpha+1)}[\psi(x)-\psi(a)]^{k-\alpha}$

Leibniz rule is an example that different fractional derivatives have different properties, usually different from the properties of natural order derivatives.

It is difficult to find a general method for solving differential equations with various fractional derivatives. For example, the subdiffusion equation with the ordinary Caputo time derivative can be solved by means of the Laplace transform method while the g–subdiffusion equation is solved using the g–Laplace transform.

Application of differential equations with fractional derivatives in physics

Two main ways to get a fractional equation

1. Equation with fractional derivative has a physical basis (the equation is derived from a physical model - an example: anomalous diffusion)
2. Fractional equation is obtained by simply replacing the derivative of natural order with a fractional order derivative. One gets a "more general" equation, but its physical interpretation is rather unknown.

Integral transforms

Fractional derivatives often take the form of a convolution of functions

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{d^n}{dt^n} \int_a^t K(t-\tau,\alpha) f(\tau) d\tau, \; \frac{d^\alpha f(t)}{dt^\alpha} = \int_a^t K(t-\tau,\alpha) \frac{d^n f(\tau)}{d\tau^n} d\tau,$$

Laplace transform

$$\hat{f}(s) \equiv \mathcal{L}[f(t)](s) = \int_0^\infty \mathrm{e}^{-st} dt, \; \mathcal{L}[(f *_t g)(t)](s) = \hat{f}(s)\hat{g}(s)$$

$$(f *_t g)(t) = \int_0^t f(t-t') g(t') dt'$$

Fourier transform

$$\tilde{f}(k) \equiv \mathcal{F}[f(x)](k) = \int_{-\infty}^\infty \mathrm{e}^{ikx} dt, \; \mathcal{F}[(f *_x g)(x)](k) = \tilde{f}(k)\tilde{g}(k)$$

$$(f *_x g)(x) = \int_{-\infty}^\infty f(x-x') g(x') dx'$$

The most commonly used fractional derivatives in physics

Caputo fractional derivative, $0 < \alpha < 1$

$$\frac{{}^{C}d^{\alpha} f(t)}{dt^{\alpha}} = \frac{1}{\Gamma(1-\alpha)} \int_0^t [t-u]^{-\alpha} f'(u) du, \quad \mathcal{L}\left[\frac{{}^{C}d^{\alpha} f(t)}{dt^{\alpha}}\right](s) = s^{\alpha}\hat{f}(s) - s^{\alpha-1} f(0)$$

Riemann–Liouville fractional derivative, $0 < \alpha < 1$

$$\frac{{}^{RM}d^{\alpha} f(t)}{dt^{\alpha}} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t [t-u]^{-1-\alpha} f(u) du, \quad \mathcal{L}\left[\frac{{}^{RL}d^{\alpha} f(t)}{dt^{\alpha}}\right](s) = s^{\alpha}\hat{f}(s)$$

Riesz–Weyl fractional derivative, $1 < \gamma < 2$

$$\frac{d^{\gamma} f(x)}{dx^{\gamma}} = \frac{1}{\Gamma(1-\gamma)} \int_{-\infty}^{x} [x-u]^{-\gamma} \frac{df(u)}{dt} du, \quad \mathcal{F}\left[\frac{d^{\gamma} f(x)}{d|x|^{\gamma}}\right](s) = -|k|^{\gamma}\tilde{f}(k)$$

Fractional Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = C(-\hbar^2\Delta)^{\alpha/2}\psi$$

with Riesz spatial fractional derivative, $\alpha \in (0, 2)$, N. Laskin, Phys. Rev. E **66**, 056108 (2002), X. Guo, M. Xum J. Math. Phys. **47**, 082104 (2006) , *application to describe fractional dark energy:* R.G. Landim, Phys. Rev. D **104**, 103508 (2021) .

$$(i\hbar)^{\alpha}\frac{\partial^{\alpha}\psi}{\partial t^{\alpha}} = \hat{H}\psi$$

with Caputo time fractional derivative, $\alpha \in (0, 1)$, M. Naber, J. Math. Phys. **45**, 3339 (2004), A. Iomin, Phys. Rev. E **80**, 022103 (2009) .

Other generalized physical equations with fractional derivatives (we do not consider here if there is a physical basis for these equations)

- fractional Dirac equation, S. Muslih et al., J. Phys. A **43**, 055203 (2010), A. Raspini, Phys. Scripta **64**, 20 (2001)
- fractional Schrödinger-Klein-Gordon equation, J. Blackledge et al., Math. Aeterna **3**, 601 (2013)
- fractional Maxwell equations, E.K. Jaradat et al., J. Math. Phys. **53**, 033505 (2012)
- fractional Liouville equation, V.E. Tarasov, Phys. Plasmas **20**, 102101 (2013)
- fractional Newtonian mechanics, D. Baleanu et al., Cent. Europ. J. Phys. **8**, 120 (2010), W.S. Chung, J. Comput. Appl. Math. **290**, 150 (2015), G.U. Varieschi, J. Appl. Math. Phys. **6**, 1247 (2018)
- fractional telegrapher's equation, J. Masoliver, Phys. Rev. E **93**, 052107 (2016)
- fractional oscillator equation, Y.E. Ryabov et al., Phys. Rev. B **66**, 184201 (2002)
- fractional Maxwell model of viscoelastic oscillator, Z.-L. Li et al., J. Vibr. Engin. Techn. **6:5**, (2018)
- . . . and many others

Normal and anomalous diffusion

Normal diffusion

$$\frac{\partial P(x,t|x_0)}{\partial t} = D\frac{\partial^2 P(x,t|x_0)}{\partial x^2}, \quad P(x,0|x_0) = \delta(x-x_0)$$

$$P(x,0|x_0) = \delta(x-x_0), \quad P(\pm\infty,t|x_0) = 0$$

$$\sigma^2(t) \equiv \int_{-\infty}^{\infty} (x-x_0)^2 P(x,t|x_0)dx = 2Dt$$

Normal and anomalous diffusion

Anomalous diffusion

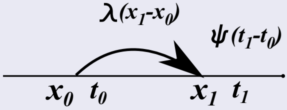
Anomalous diffusion is a process that is not normal diffusion. Important functions that characterize the anomalous diffusion process are different from those of normal diffusion.

$$\sigma^2(t) \sim \begin{cases} t^{\beta}, \ \beta > 1, \ \text{for superdiffusion}, \\ t, \ \text{for normal diffusion}, \\ t^{\alpha}, \ 0 < \alpha < 1, \ \text{for subdiffusion}, \\ f(\log t), \ \text{for slow subdiffusion} \end{cases}$$

Qualitatively different diffusion processes

- **Superdiffusion** *(facilitated diffusion) occurs in media where rapid movement of molecules over long distances is common*. Examples: diffusion in turbulent media, epidemic spread (transmission of viruses by air travel), movement of endogeneous intracellular particles in some pathogens, of soil amebas on plastic or glass surfaces in liquid media, mussels movement, cell migration in some biological processes, diffusion in random velocity fields.
- **Normal diffusion** Examples: the most known cases of diffusion, eg diffusion of various substances in water.
- **Ordinary subdiffusion** *occurs in media in which the movement of particles is very hindered*. Examples: transport of some molecules in viscoelastic chromatin network, porous media, living cells, transport of sugars in agarose gel, transport of antibiotic in bacterial biofilm.
- **Slow subdiffusion (ultraslow diffusion)** *(hindered subdiffusion)* Examples: transport of water in aqueous sucrose glasses, language dynamics, diffusion in very crowded media.

Normal and anomalous diffusion



$\langle\lambda^2(x)\rangle < \infty$ $\langle\psi(t)\rangle < \infty$	$\langle\lambda^2(x)\rangle < \infty$ $\langle\psi(t)\rangle = \infty$	$\langle\lambda^2(x)\rangle = \infty$ $\langle\psi(t)\rangle < \infty$
normal diffusion	ordinary subdiffusion	superdiffusion
$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2}$ $\lambda(x) \sim e^{-x^2/2\sigma}$ $\psi(t) \sim e^{-t/\tau}$	$\frac{\partial C}{\partial t} = D_\alpha \frac{\partial_{RL}^{1-\alpha}}{\partial t^{1-\alpha}} \frac{\partial^2 C}{\partial x^2}$ $\lambda(x) \sim e^{-x^2/2\sigma}$ $\psi(t) \sim \left(\frac{\tau}{t}\right)^{1+\alpha}, \; t \gg \tau, \; 0 < \alpha < 1$	$\frac{\partial C}{\partial t} = D_\beta \frac{\partial^\beta C}{\partial x^\beta}$ $\lambda(x) \sim \sigma^{-\beta} \, \lvert x\rvert^{-1-\beta}$ $\lvert x\rvert \gg \sigma, \; 1 < \beta < 2$ $\psi(t) \sim e^{-t/\tau}$

Slow subdiffusion

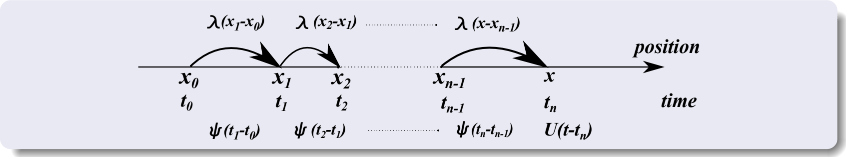
$$\frac{\partial C(x,t)}{\partial t} = \int_0^t K(\log(t'))\frac{\partial^2 C(x,t-t')}{\partial x^2}$$

slow sub. $\langle t^\rho\rangle = \infty$ for $\rho > 0$

ordinary sub. $\langle t^\rho\rangle = \infty$ for $\rho > \alpha$

Continuous time random walk (CTRW) model

E.W. Montroll, G.H. Weiss, J. Math. Phys. **6**, 167 (1965), A. Compte, Phys. Rev. E **53**, 4191 (1996), R. Metzler, J. Klafter, Phys. Rep. **339**, 1 (2000), I.M. Sokolov, J. Klafter, Chaos **15**, 026103 (2005), E. Barkai et al., Phys. Rev. E **61**, 132 (2000) and many others



$$P(x,t|x_0) = \sum_{n=0}^{\infty} Q_n(t) P_n(x|x_0)$$

$Q_n(t)$ is a probability that a diffusing particle takes n step in the time interval $(0,t)$, $P_n(x|x_0)$ is a probability density that the particle will be at x after making n jumps, x_0 is the initial particle position, U is the probability that a particle does not change its position after the last jump, $U(t) = 1 - \int_0^t \psi(t')dt'$

$$Q_n(t) = (\underbrace{\psi *_t \psi *_t \ldots *_t \psi}_{n \text{ times}} *_t U)(t), \quad P_n(x|x_0) = (\underbrace{\lambda *_x \lambda *_x \ldots *_x \lambda}_{n \text{ times}})(x),$$

$$P(x,t|x_0)=\sum_{n=0}^{\infty}Q_n(t)P_n(x|x_0)$$

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$$(f *_t h)(t)=\int_0^t f(t')h(t-t')dt',\ \ \mathcal{L}[(f *_t h)(t)](s)=\mathcal{L}[f(t)](s)\mathcal{L}[h(t)](s)\equiv\hat{f}(s)\hat{g}(s)$$

$$\hat{U}(s)=\mathcal{L}[1-\int_0^t\psi(t')dt']=\frac{1-\hat{\psi}(s)}{s}$$

$$(f *_x h)(x)=\int_{-\infty}^{\infty} f(x')h(x-x')dx',\ \ \mathcal{F}[(f *_t h)(x)](k)=\mathcal{F}[f(x)](k)\mathcal{F}[h(x)](k)\equiv\tilde{f}(k)\tilde{g}(k)$$

$$\hat{\tilde{P}}(k,s)=\frac{1-\hat{\psi}(s)}{s}\sum_{n=0}^{\infty}[\hat{\psi}(s)\tilde{\lambda}(k)]^n=\frac{1-\hat{\psi}(s)}{s}\frac{1}{[1-\hat{\psi}(s)\tilde{\lambda}(k)]}$$

$\hat{\psi}(s) = \int_0^\infty \exp(-st)\psi(t)dt$, $\tilde{\lambda}(k) = \int_{-\infty}^\infty \exp(ikx)\lambda(x)dx$

$\exp(u) = \sum_{j=0}^\infty u^j/j!$,

$\left\langle t^j \right\rangle = \int_0^\infty t^j\psi(t)dt$, $\left\langle x^j \right\rangle = \int_{-\infty}^\infty x^j\lambda(x)dx$,

$$\tilde{\lambda}(k) = \sum_{j=0}^{\infty} (ik)^j \frac{\left\langle x^j \right\rangle}{j!} \ , \ \hat{\psi}(s) = \sum_{j=0}^{\infty} (-s)^j \frac{\left\langle t^j \right\rangle}{j!} \ .$$

$$\langle t \rangle = - \left. \frac{d\hat{\psi}(s)}{ds} \right|_{s=0} , \ \left\langle x^2 \right\rangle = - \left. \frac{d^2\tilde{\lambda}(|k|)}{d|k|^2} \right|_{k=0}$$

$$\hat{\psi}(s) = 1 - \tau s + \tau^2 s^2/2 - \dots,\ \langle t \rangle = - \left. \frac{d\hat{\psi}(s)}{ds} \right|_{s=0} = \tau$$

$$\tilde{\lambda}(k) = 1 - \rho^2 |k|^2/2 + \dots,\ \left\langle x^2 \right\rangle = - \left. \frac{d^2\tilde{\lambda}(k)}{dk^2} \right|_{k=0} = \rho^2$$

$$\hat{\psi}(s) = 1 - \tau s^\alpha + \tau^2 s^{2\alpha}/2 - \dots,\ \langle t \rangle = - \left. \frac{d\hat{\psi}(s)}{ds} \right|_{s=0} = \left. \frac{\tau}{s^{1-\alpha}} \right|_{s=0} = \infty,\ 0 < \alpha < 1$$

$$\tilde{\lambda}(|k|) = 1 - \rho^2 |k|^\gamma/2 + \dots,\ \left\langle x^2 \right\rangle = - \left. \frac{d^2\tilde{\lambda}(|k|)}{d|k|^2} \right|_{k=0} = \left. \frac{\rho^2\gamma(\gamma-1)}{|k|^{2-\gamma}} \right|_{k=0} = \infty,\ 1 < \gamma < 2$$

$$\hat{\tilde{P}}(k,s) = \frac{1-\hat{\psi}(s)}{s} \frac{1}{[1-\hat{\psi}(s)\tilde{\lambda}(k)]}$$

$$\hat{\psi}(s) = 1 - \tau s^\alpha,\ s \to 0;\ \ \tilde{\lambda}(|k|) = 1 - \rho^2|k|^2/2,\ |k| \to 0,\ D_\alpha = \rho^2/2\tau$$

$$s^\alpha \hat{\tilde{P}}(k,s) - s^{\alpha-1}P(x,0) = -D_\alpha |k|^2 \hat{\tilde{P}}(k,s),\ \ \frac{{}^C\partial^\alpha P(x,t)}{\partial t^\alpha} = D_\alpha \frac{\partial^2 P(x,t)}{\partial x^2}$$

$$\hat{\psi}(s) = 1 - \tau s,\ s \to 0;\ \ \tilde{\lambda}(|k|) = 1 - \rho^\gamma|k|^\gamma/2,\ |k| \to 0,\ D_\gamma = \rho^\gamma/2\tau$$

$$s^\alpha \hat{\tilde{P}}(k,s) - s^{\alpha-1}P(x,0) = -D_\alpha |k|^\gamma \hat{\tilde{P}}(k,s),\ \ \frac{\partial P(x,t)}{\partial t} = D_\gamma \frac{\partial^\gamma P(x,t)}{\partial x^\gamma}$$

Limitations of anomalous diffusion models

- Anomalous diffusion equations with constant parameters can only describe the diffusion of molecules in a homogeneous medium that does not change its properties over time. However, in many processes there are changes in the structure of the medium. If the changes are relatively small, only the diffusion parameters change. If the changes are large, there may be a change in diffusion type.
- The fractional superdiffusion equation is non-local in space. Thus, no boundary conditions are imposed at partially or fully reflecting walls for this equation. This equation is not used to describe superdiffusion in a membrane system. In general, the use of this equation to describe superdiffusion processes is weak. Superdiffusion models are mainly based on simulations.

The aim: to find an equation that describes each type of diffusion. This model can describe different types of diffusion processes in a system with parameters evolving over time, even when the diffusion type changes. Boundary conditions can be assumed even when the equation describes superdiffusion.

The type of diffusion depends on the interaction of diffusing molecules with the environment and on a structure of the medium, both can change over time. Anomalous diffusion with evolving anomalous diffusion exponent has been observed in:

- endogenous lipid granules in living yeast cells (J.H. Jeon *et al.*, Phys. Rev. Lett. **106**, 048103 (2011))
- microspheres in a living eukaryotic cell (A. Caspi *et al.*, Phys. Rev. Lett. **85**, 011916 (2002))
- in bacterial motion on small beads in a freely suspended soap film (X.L. Wu, A. Libchaber, Phys. Rev. Lett. **84**, 3017 (2000))
- transport of colloidal particles between two parallel plates (A. Chakrabarty *et al.*, Phys. Rev. Lett. **111**, 160603 (2013))
- bacterial defense abilities evolve over time, which can change diffusion parameters (T. Kosztołowicz, R. Metzler, Phys. Rev. E **102**, 032408 (2020); T. Kosztołowicz, R. Metzler, S. Wąsik, M. Arabski, PLoS One **15**(12), e0243003 (2020))

A change in the diffusion type can occur in:

- diffusion of passive molecules in the active bath where moving particles can affect the movement of passive molecules, active swimmers can enhance diffusion of passive particles (G. Miño *et al.*, Phys. Rev. Lett. **106**, 048102 (2011))
- diffusion of self-propelled particles and passive particles in an environment with motile microorgamisms (C. Bechinger *et al.*, Rev. Mod. Phys. **88**, 045006 (2016)). Active molecules can take energy from the environment and use it to make long jumps. Some diffusing molecules can use chemical reactions to achieve autonomous propulsion (J.R. Howse *et al.*, Phys. Rev. Lett. **99**, 048102 (2007)). This mechanism leads to the process in which σ^2 evolves much faster than the linear function of time.

- In intracellular transport in most eukaryotic cells molecules diffuse through the filament network. When particle transport is carried out along filaments, the particles can move “superdiffusionally”. Changing the orientation and polarization of the filaments may change the nature of diffusion (D. Ando *et al.*. Biophys. J. **109**, 1574 (2015)) .
- Some microorganisms, such as bacteria, move more quickly in more viscous media. This is because the addition of a viscosity enhancer creates a quasi-rigid network to facilitate the transport of molecules (H.C. Berg, L. Turner, Nature **278**, 349 (1979); Y. Magariyama, S. Kudo, Biophys. J. **83**, 733 (2002)) . Thus, an increase in viscosity may paradoxically facilitate diffusion.
- Various bacterial defense mechanisms against the action of an antibiotic may hinder but also facilitate the diffusion of antibiotic molecules in the biofilm (G.G. Anderson, G.A. O'Toole, Bacterial Biofilms, Current Topics in Microbiology and Immunology **322**, p. 85 (Berlin, Springer, 2008); T.F.C. Mah, G.A. O'Toole, Trends Microbiol. **9**, 34 (2001)) .

New Idea - g–subdiffusion equation

Ordinary Caputo fractional derivative $\frac{^C d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-u)^{-\alpha} f'(u)du$, $\alpha \in (0,1)$. G–Caputo derivative with respect to another function g, R. Almeida, Commun. Nonlinear Sci. Numer. Simul. **44**, 460 (2017)

$$\frac{^C d_g^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t [g(t)-g(u)]^{-\alpha} f'(u)du$$

$g(0)=0$, $g(\infty)=\infty$, $g'(t)>0$ for $t>0$

$$\frac{^C d_g f(t)}{dt} = \lim_{\alpha\to 1^-} \frac{^C d_g^\alpha f(t)}{dt^\alpha} = \frac{f'(t)}{g'(t)}$$

G–Laplace transform $\mathcal{L}_g[f(t)](s) = \int_0^\infty \mathrm{e}^{-sg(t)} f(t) g'(t) dt$, F. Jarad et al., Discrete Contin. Dyn. Syst., Ser s **13**, 709 (2020), Adv. Differ. Eqn. **2020**, 72 (2020)

The g–Laplace transform is related to the ordinary Laplace transform as follows

$$\mathcal{L}_g[f(t)](s) = \mathcal{L}[f(g^{-1}(t))](s)$$

$$\mathcal{L}_g[f(t)](s) = \mathcal{L}[h(t)](s) \Leftrightarrow f(t) = h(g(t))$$

Fractional g–subdiffusion equation

$$\frac{{}^{C}\partial_g^{\alpha} P_g(x,t|x_0)}{\partial t^{\alpha}} = D\frac{\partial^2 P_g(x,t|x_0)}{\partial x^2}$$

$$\begin{aligned}\mathcal{L}_g\left[\frac{{}^{C}d_g^{\alpha} f(t)}{dt^{\alpha}}\right](s) &= s^{\alpha}\mathcal{L}_g[f(t)](s) - s^{\alpha-1}f(0)\\ \mathcal{L}\left[\frac{{}^{C}d^{\alpha} f(t)}{dt^{\alpha}}\right](s) &= s^{\alpha}\mathcal{L}[f(t)](s) - s^{\alpha-1}f(0)\end{aligned}$$

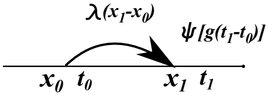
$$s^{\alpha}\mathcal{L}_g[C_g(x,t)](s) - s^{\alpha-1}C_g(x,0) = D\frac{\partial^2\mathcal{L}_g[C_g(x,t)](s)}{\partial x^2}$$

$$s^{\alpha}\mathcal{L}[C_{\alpha}(x,t)](s) - s^{\alpha-1}C_{\alpha}(x,0) = D\frac{\partial^2\mathcal{L}[C_{\alpha}(x,t)](s)}{\partial x^2}$$

$$\mathcal{L}_g[C_g(x,t)](s) = \mathcal{L}[C_{\alpha}(x,t)](s) \Leftrightarrow C_g(x,t) = C_{\alpha}(x,g(t))$$

Stochastic interpretation

T. Kosztołowicz and A. Dutkiewicz, *Stochastic interpretation of g-subdiffusion process*, Phys. Rev. E **104**, L042101 (2021)



g–CTRW, $\mathcal{L}[f(t)](s) \to \mathcal{L}_g[f(t)](s)$,
$(f *_t h)(t) \to (f *_g h)(t) = \int_0^t f(u)[g^{-1}(g(t)-g(u))]g'(u)du$

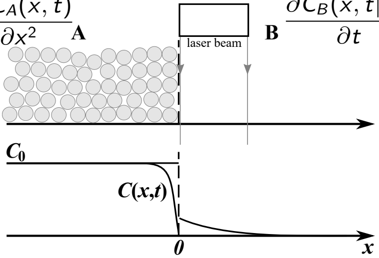
$$\mathcal{L}_g[(f *_g h)(t)](s) = \mathcal{L}_g[f(t)](s)\mathcal{L}_g[h(t)](s)$$

T. Kosztołowicz, A. Dutkiewicz, K. D. Lewandowska, S. Wąsik, and M. Arabski, *Subdiffusion equation with Caputo fractional derivative with respect to another function in modelling diffusion in a complex system consisting of matrix and channels*, Phys. Rev. E **106**, 044138 (2022).

Anomalous diffusion of antibiotic (colistin) in a system consisting of packed gel (alginate) beads immersed in water was studied experimentally and theoretically. The experimental studies were performed using the interferometric method of measuring concentration profiles of diffusing substance.

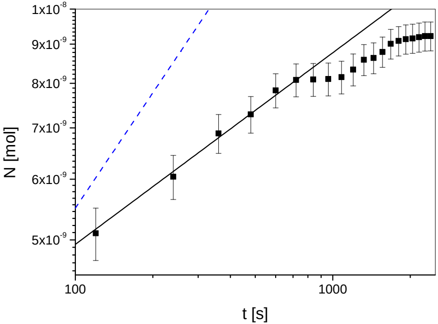
$$\frac{{}^{C}\partial_g^{\alpha} C_A(x,t)}{\partial t^{\alpha}} = D_A \frac{\partial^2 C_A(x,t)}{\partial x^2}$$

$$\frac{\partial C_B(x,t|x_0)}{\partial t} = D_B \frac{\partial^2 C_B(x,t)}{\partial x^2}$$



$$N(t) = \int_{-\infty}^{0} [C_0 - C_A(x,t)]dx = \frac{C_0\sqrt{D}}{\Gamma(1+\alpha/2)} g^{\alpha/2}(t)$$

$$N(t) = \kappa t^{\tilde{\alpha}(t)/2}, \quad \tilde{\alpha}(t) = \frac{\alpha}{1+\beta t}, \quad g(t) = \kappa^{2/\alpha} t^{1/(1+\beta)}$$



Plots of the power functions $N(t) = 1.57 \times 10^{-9} t^{0.25}$ for "ordinary" subdiffusion (solid line) and $N(t) = 0.55 \times 10^{-9}\sqrt{t}$ for normal diffusion (dashed line) in the log–log scale, the empirical results are denoted by symbols. The function for normal diffusion is an example.

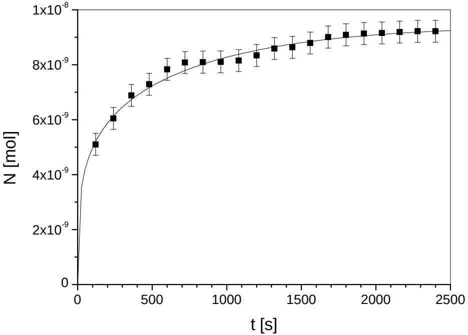


Figure: Plot of the function N with $\tilde{\alpha}(t)$ (solid line) for $\kappa = 1.57 \times 10^{-9}\ \mathrm{mol/s}^{0.25}$, $\alpha = 0.5$, and $\beta = 4.3 \times 10^{-5}\ 1/\mathrm{s}$, the empirical results are denoted by symbols.

T. Kosztołowicz, *Subdiffusion equation with fractional Caputo time derivative with respect to another function in modeling transition from ordinary subdiffusion to superdiffusion*, arXiv:2210.11346.

Ordinary subdiffusion equation

$$\frac{{}^C\partial^\alpha P_\alpha(x,t|x_0)}{\partial t^\alpha} = D_\alpha \frac{\partial^2 P_\alpha(x,t|x_0)}{\partial x^2},\ \alpha \in (0,1)$$

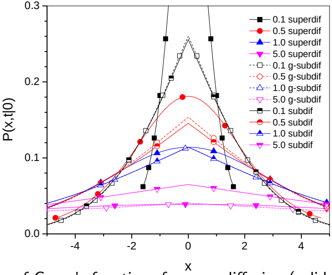
Superdiffusion equation

$$\frac{\partial P_\gamma(x,t|x_0)}{\partial t} = D_\gamma \frac{\partial^\gamma P_\gamma(x,t|x_0)}{\partial x^\gamma},\ \gamma \in (1,2)$$

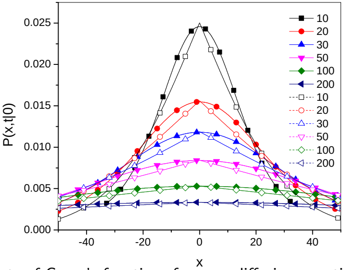
Transition form ordinary subdiffusion to superdiffusion

$$\frac{{}^C\partial_g^\alpha P_g(x,t|x_0)}{\partial t^\alpha} = D_\alpha \frac{\partial^2 P_g(x,t|x_0)}{\partial x^2}$$

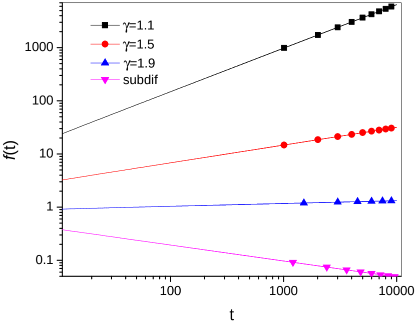
$$g(t) = a(t)t + [1 - a(t)]Et^{\frac{2}{\gamma\alpha}},\ a(0) = 1,\ a(\infty) = 0,\ \ a(t) = \frac{1}{1 + At^\nu}$$



Plots of Green's functions for superdiffusion (solid lines with filled symbols) with $\gamma = 1.5$ and $D_\gamma = 4$, for ordinary subdiffusion (solid lines with half–filled symbols) with $\alpha = 0.7$ and $D_\alpha = 10$, and for g–subdiffusion (dashed lines with open symbols), $A = 1$, $\nu = 1.2$, and α, D_α given above. The plots are made for times given in the legend.



Plots of Green's functions for superdiffusion equation (solid lines with filled symbols), and for g–subdiffusion equation (dashed lines with open symbols) for times given in the legend, the values of the parameters are the same as in the caption of Fig. 1.



Time evolution of the jump frequency $f(t) = d\langle n(t)\rangle/dt$. For ordinary subdiffusion f is an decreasing function of time, the superdiffusion effect is achieved when f is a function increasing with time.

Other applications of the g–subdiffusion equation

- Continuous transition from ordinary subdiffusion to slow subdiffusion: T. Kosztołowicz and A. Dutkiewicz, *Subdiffusion equation with Caputo fractional derivative with respect to another function*, Phys. Rev. E **104**, 014118 (2021)
- Ordinary subdiffusion with a change in values of subdiffusion parameters: T. Kosztołowicz and A. Dutkiewicz, *Composite subdiffusion equation that describes transient subdiffusion*, Phys. Rev. E **106**, 044119 (2022)
- Subdiffusion with absorption of molecules with variable subdiffusion and absorption parameters (including oscillatory effects of changing parameters): T. Kosztołowicz, *First passage time for the g-subdiffusion process of vanishing particles, Phys. Rev. E* **106**, L022104 (2022)

Final remarks

Universality of the g–subdiffusion equation

- The g–subdiffusion equation can describe superdiffusion, normal diffusion, ordinary subdiffusion, slow subdiffusion.
- It describes a continuous transition between these processes.
- It describes processes with evolving parameter values.
- For each of these processes, boundary conditions can be derived at a partially permeable wall (including superdiffusion).

Hypothesis. It seems that the g–subdiffusion equation can be treated as the unified anomalous diffusion equation.

Open question

Can fractional calculus help develop a unified model of other physical processes?