# Stability of Classical Chromodynamic Fields

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PhD seminar

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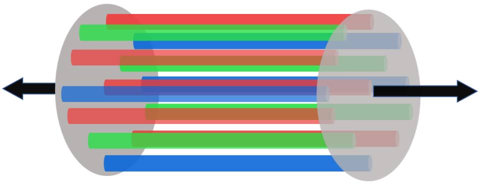
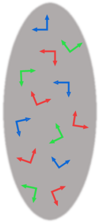
# Overview

- Motivation
- Linearized Classical Chromodynamics
- Stability analysis in Minkowski coordinates
- Summary
- Curvilinear coordinates
- Stability analysis in Milne coordinates

## Motivation

- The earliest phase of heavy-ion collisions is described in terms of classical fields.

- Early configuration is unstable, but the character of the instabilities is not clear (P. Romatschke and R. Venugopalan, Phys. Rev. Lett. 96, 062302 (2006)).
- We plan to study the problem systematically.

## Yang-Mills equations & linearized QCD

### Yang-Mills equations in adjoint representation

$$D_\mu^{ab} F_b^{\mu\nu} = \partial_\mu F_a^{\mu\nu} + g f^{abc} A_\mu^b F_c^{\mu\nu} = J_a^\nu, \qquad F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g f^{abc} A_b^\mu A_c^\nu$$

### Linearized QCD

$A_a^\mu(t,\mathbf{r}) = \bar{A}_a^\mu(t,\mathbf{r}) + a_a^\mu(t,\mathbf{r}),$ where $|\bar{A}(t,\mathbf{r})| \gg |a(t,\mathbf{r})|$

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## Background gauge condition

$$\bar{D}_{ab}^\mu a_\mu^a = \partial^\mu a_\mu^a + g f^{abc} \bar{A}_b^\mu a_\mu^c = 0$$

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$$\bar{D}_{ab}^\mu a_\mu^a = \partial^\mu a_\mu^a + g f^{abc} \bar{A}_b^\mu a_\mu^c = 0$$

## Linearized Yang-Mills equations in the background gauge

$$\left[ g^{\mu\nu} \left( \bar{D}_\rho \bar{D}^\rho \right)_{ac} + 2 g f^{abc} \bar{F}_b^{\mu\nu} \right] a_\nu^c = J_a^\mu$$

# Uniform chromoelectric and chromomagnetic fields & assumpions

Generation of constant fields in one direction is possible in two ways:

- Abelian configuration - single color potential linearly dependent on coordinates,
- nonAbelian configuration - multicolor, constant potential.

Calculations done in SU(2) group: $f^{abc} \to \epsilon^{abc}$.

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We consider 5 configurations:

- Abelian and nonAbelian configurations of chromomagnetic field,
- Abelian and nonAbelian configuration of chromoelectric field,
- Abelian configuration of both chromomagnetic and chromoelectric fields together.

## Abelian configuration of chromomagnetic field

### Constant homogeneous chromomagnetic field

$\bar{A}^{\mu}_{a}(t, \mathbf{r}) = (0, 0, xB, 0)\delta^{a1}$

Potential $\bar{A}^{\mu}_{a}(t, \mathbf{r})$ satisfies YM equations with vanishing current

$a^{\mu}_{a}(t, x, y, z) = e^{-i(\omega t - k_y y - k_z z)} a^{\mu}_{a}(x)$ & mixing of colors and coordinates

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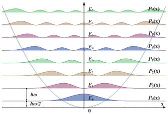
Non-relativistic Schrödinger equation of harmonic oscillator

$\Big( -2m\mathcal{E} + m^2\bar{\omega}^2(x_0 - x)^2 - \frac{d^2}{dx^2} \Big)\varphi(x) = 0$



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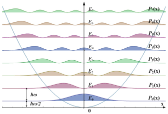
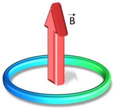


## Unstable solution

$\omega_-^2 = k_z^2 - gB < 0$ for $gB > k_z^2 \quad \longrightarrow \quad a \sim e^{\sqrt{gB - k_z^2}\,t}$

Nielsen Olesen instability

The result is purely classical!



# nonAbelian configuration of chromomagnetic field

## Constant homogeneous chromomagnetic field

$$\bar{A}_a^\mu = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{B/g} \\ 0 & 0 & \sqrt{B/g} & 0 \end{bmatrix}, \quad J_a^\nu = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{gB^3} \\ 0 & 0 & \sqrt{gB^3} & 0 \end{bmatrix}.$$

$a_a^\mu(t, \mathbf{x}) = a_a^\mu e^{-i(\omega t - \mathbf{kx})}$

## Matrix equations

12x12 matrix in block form $\longrightarrow$ 2 equal matrices 3x3 and one 6x6

Homogeneous equations $\longrightarrow$ solutions exist if determinant of the matrix vanishes.

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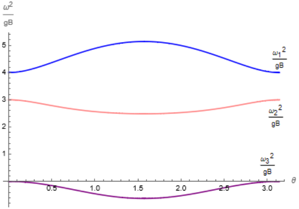
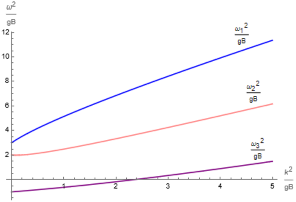



$\frac{\omega_n^2}{gB}$ as a function of $\theta$ for $\mathbf{k}^2 = gB$




$\frac{\omega_n^2}{gB}$ as a function of $\mathbf{k}^2$ for $\theta = \frac{\pi}{2}$

# Abelian configuration of chromoelectric field

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$\bar{A}^{\mu}_{a}(t, \mathbf{r}) = (-zE, 0, 0, 0)\delta^{a1}$

Potential $\bar{A}^{\mu}_{a}(t, \mathbf{r})$ satisfies YM equations with vanishing current

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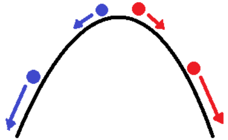
$$\Big(-g^2 z^2 E^2 + k_x^2 + k_y^2 - \frac{d^2}{dz^2}\Big)Y^{\pm}(z) = 0$$

The equation coincides with the non-relativistic Schrödinger equation of inverted harmonic oscillator



## Solutions

run-away solutions $\longrightarrow$ unstable

## nonAbelian configuration of chromoelectric field

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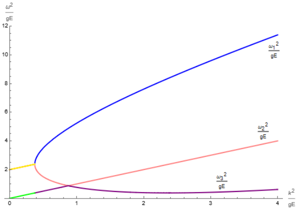
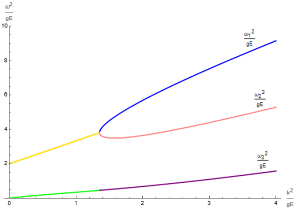



$\frac{\omega_n^2}{gE}$ as a function of $\frac{k^2}{gE}$ for $\theta = 0$

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# Abelian configuration of chromomagnetic and chromoelectric fields

The configuration of parallel chromoelectric and chromomangetic fields can be generated only in Abelian way.

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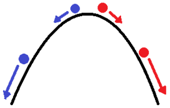
The chromoelectric field determines the system's behaviour



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## Gauge dependence

Energy-Momentum Tensor is gauge invariant $T^{\mu\nu} = F_a^{\mu\rho} F_\rho{}^\nu{}_a + \frac{1}{4} g^{\mu\nu} F_a^{\sigma\tau} F_{\sigma\tau\, a}$

### The diagonal elements

- energy density $\varepsilon = T^{00} = \frac{1}{2}(\mathbf{E}_a \cdot \mathbf{E}_a + \mathbf{B}_a \cdot \mathbf{B}_a)$
- longitudinal pressure $p_L = T^{zz} = -E_a^z E_a^z - B_a^z B_a^z + \varepsilon$
- transverse pressure $p_T = T^{xx} = -E_a^x E_a^x - B_a^x B_a^x + \varepsilon$

### Gauge transformations

- $F^{\mu\nu} \longrightarrow U F^{\mu\nu} U^\dagger$
- $A^\mu = \bar{A}^\mu + a^\mu \longrightarrow U \bar{A}^\mu U^\dagger + U a^\mu U^\dagger + \frac{i}{g} U \partial^\mu U^\dagger$

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- $\bar{A}^\mu \longrightarrow U \bar{A}^\mu U^\dagger + \frac{i}{g} U \partial^\mu U^\dagger, \qquad a^\mu \longrightarrow U a^\mu U^\dagger$

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- $\bar{A}^\mu \longrightarrow U \bar{A}^\mu U^\dagger + \frac{i}{g} U \partial^\mu U^\dagger, \qquad a^\mu \longrightarrow U a^\mu U^\dagger$

The strength tensor with neglected terms quadratic in $a^\mu$ still transforms as $F^{\mu\nu} \longrightarrow U F^{\mu\nu} U^\dagger$.

## Energy density and pressure for stable and unstable modes

We considered modes from the Abelian configuration of chromomagnetic field.

### Stable mode

$$\varepsilon = \frac{1}{2}B^2 + gB\delta^2(3 + 2gx^2B)e^{-gBx^2}$$

$$p_L = -\frac{1}{2}B^2 + gB\delta^2(3 - 2gx^2B)e^{-gBx^2}$$

$$p_T = \frac{1}{2}B^2 + 2g^2x^2B^2\delta^2e^{-gBx^2}$$

### Unstable mode

$$\varepsilon = \frac{1}{2}B^2 + gB\delta^2e^{-gBx^2}e^{2\sqrt{gB}t}$$

$$p_L = -\frac{1}{2}B^2 + gB\delta^2e^{-gBx^2}e^{2\sqrt{gB}t}$$

$$p_T = \frac{1}{2}B^2$$

# Summary

- We found complete spectra of eigenmodes for Abelian and nonAbelian constant and uniform chromoelectric and chromomagnetic fields.
- The spectra of Abelian and nonAbelian fields configurations are rather different, but there are everywhere unstable modes.
- The Energy-Momentum Tensor ($T^{\mu\nu}$) with fields linearized in $a_a^\mu$ is gauge invariant.

S.Bazak, S.Mrówczyński, Stability of classical chromodynamic fields, Phys. Rev. D 105, 034023 (2022)
& S.Bazak, S.Mrówczyński, Addendum to "Stability of classical chromodynamic fields", Phys. Rev. D 106, 034031 (2022)

## Yang-Mills equations in Milne coordinates - progress report

Yang-Mills fields in Color Glass Condensate are a boost invariant system.
The most natural tool one can use to describe boost invariant configurations are Milne coordinates.

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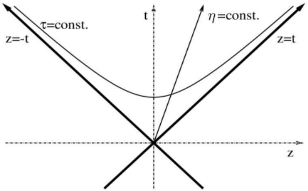
The most natural tool one can use to describe boost invariant configurations are Milne coordinates.

Milne coordinates: $\tilde{x}^\mu = (\tau, x, y, \eta)$ are defined

$$\begin{cases} \tau = \sqrt{t^2 - z^2}, \\ \eta = \frac{1}{2}\ln\frac{t+z}{t-z}, \end{cases} \qquad \begin{cases} t = \tau\cosh\eta, \\ z = \tau\sinh\eta. \end{cases}$$




The transformation matrices
-for contravariant four-vectors -for covariant four-vectors

$$M^\mu{}_\nu = \frac{d\tilde{x}^\mu}{dx^\nu} = \begin{bmatrix} \cosh\eta & 0 & 0 & -\sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\sinh\eta}{\tau} & 0 & 0 & \frac{\cosh\eta}{\tau} \end{bmatrix}, W_\mu{}^\nu = \frac{dx^\nu}{d\tilde{x}^\mu} = \begin{bmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \tau\sinh\eta & 0 & 0 & \tau\cosh\eta \end{bmatrix}.$$

# Curvilinear coordinates - Milne coordinates

The $\tilde{x}^{\mu} = (\tau, x, y, \eta)$ does not form a four-vector.

$$M^{\mu}{}_{\nu} x^{\nu} = \begin{bmatrix} \cosh\eta & 0 & 0 & -\sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\sinh\eta}{\tau} & 0 & 0 & \frac{\cosh\eta}{\tau} \end{bmatrix} \cdot \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \tau \\ x \\ y \\ 0 \end{bmatrix} = \tilde{x}^{\mu}.$$

$$W_{\mu}{}^{\nu} x_{\nu} = \begin{bmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \tau\sinh\eta & 0 & 0 & \tau\cosh\eta \end{bmatrix} \cdot \begin{bmatrix} t \\ -x \\ -y \\ -z \end{bmatrix} = \begin{bmatrix} \tau \\ -x \\ -y \\ 0 \end{bmatrix} = \tilde{x}_{\mu}.$$

## Metric Tensor

The metric tensor in Minkowski coordinates

$$g^{\mu\nu} = g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The metric tensors in Milne coordinates

$$\tilde{g}^{\mu\nu} = M^{\mu}{}_{\rho} M^{\nu}{}_{\sigma} g^{\rho\sigma} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\frac{1}{\tau^2} \end{bmatrix},$$

$$\tilde{g}_{\mu\nu} = W_{\mu}{}^{\rho} W_{\nu}{}^{\sigma} g_{\rho\sigma} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\tau^2 \end{bmatrix}.$$

## Covariant derivative

In curvilinear coordinates, differentials of a vector do not form a vector ($dA_\mu$ is not a vector) and similarly derivatives of a vector with respect to the coordinate do not form a tensor ($\partial_\mu A_\nu$ is not a tensor). This is due to fact that at different space-points vectors transform in a different way (coefficients of transformation are functions of the coordinates).

$$A_\mu = \frac{\partial x'^\nu}{\partial x^\mu} A'_\nu,$$

so

$$dA_\mu = \frac{\partial x'^\nu}{\partial x^\mu} dA'_\nu + A'_\nu \frac{\partial^2 x'^\nu}{\partial x^\mu \partial x^\sigma} dx^\sigma.$$

One defines the parallel translation of vector for which in Minkowski coordinates the coordinates of vector do not change, but in curvilinear coordinates they change

$$DA^\mu = dA^\mu - \delta A^\mu$$

$DA^\mu$ transforms as a vector, so one finds

$$\delta A^\mu = -\Gamma^\mu_{\nu\sigma} A^\nu dx^\sigma,$$

where $\Gamma^\mu_{\nu\sigma}$ are Christoffel symbols.

## Christoffel symbols $\Gamma^{\sigma}_{\mu\nu}$

$$\Gamma^{\sigma}_{\mu\nu} \equiv \frac{1}{2} g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}).$$

### Non-zero Christoffel symbols for Milne coordinates

$$\Gamma^{\tau}_{\eta\eta} = \tau, \qquad \Gamma^{\eta}_{\tau\eta} = \Gamma^{\eta}_{\eta\tau} = \frac{1}{\tau}.$$

### The covariant derivative of nonAbelian theory

acting on four-vector

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + ig[A_\mu, A^\nu] + \Gamma^{\nu}_{\mu\rho} A^\rho,$$

and acting on tensor

$$\nabla_\mu F^{\sigma\rho} = \partial_\mu F^{\sigma\rho} + \Gamma^{\sigma}_{\nu\mu} F^{\nu\rho} + \Gamma^{\sigma}_{\nu\mu} F^{\rho\nu} - ig[A_\mu, F^{\rho\sigma}],$$

## Yang-Mills equations in Milne coordinates

### YM equations

$$\frac{1}{\tau}\partial_\mu\left[\tau g^{\mu\rho}g^{\nu\sigma}\left(\partial_\rho A^a_\sigma - \partial_\sigma A^a_\rho + gf^{abc}A^b_\rho A^c_\sigma\right)\right]$$

$$+gf^{abc}A^b_\mu g^{\mu\rho}g^{\nu\sigma}\left(\partial_\rho A^c_\sigma - \partial_\sigma A^c_\rho + gf^{cde}A^d_\rho A^e_\sigma\right) = J^\mu_a$$

### Background gauge condition

$$\bar{D}^{ac}_\mu a^\mu_c = \partial_\mu a^\mu_a + gf^{abc}\bar{A}^b_\mu a^\mu_c + \Gamma^\mu_{\mu\rho}a^\rho_a = 0,$$

### Relation between fields in Minkowski and Milne coordinates

$$A^\tau = \cosh\eta\ A^t - \sinh\eta\ A^z,$$

$$A^\eta = -\frac{\sinh\eta}{\tau}\ A^t + \frac{\cosh\eta}{\tau}\ A^z.$$

*Thank you for attention!*